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Summation of Series

Some Important Formulae:

1) $a + ar + ar^2 + \cdots\cdots ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$ G. Series

2) $a + ar + ar^2 + \cdots\cdots \infty = \dfrac{a}{1-r}$ $|r|<1$ & $n\to\infty$ Infinite G. Series

3) $1 - \frac{1}{2}x + \frac{1.3}{2.4}x^2 - \frac{1.3.5}{2.4.6}x^3 \cdots \infty = \dfrac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}}$ (B. Series, $n = -\frac{1}{2}$)

4) $1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5.x^3}{2.4.6} + \cdots\infty = \dfrac{1}{\sqrt{1-x}} = (1-x)^{-\frac{1}{2}}$ (B. series, $n = -\frac{1}{2}$)

5) $x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots\cdots \infty = \sin x$

6) $1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \frac{x^8}{\lfloor 8} \cdots\cdots \infty = \cos x$

7) $1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\cdots \infty = e^x$

8) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \cdots\cdots \infty = \ln(1+x)$

9) $-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \cdots\cdots \infty = \ln(1-x)$

10) $1 + \frac{x}{2} - \frac{1.x^2}{2.4} + \frac{1.3}{2.4.6}x^3 - \frac{1.3.5}{2.4.6.8}x^4 \cdots \infty = \sqrt{1+x}$ (B Series, $n = +\frac{1}{2}$)

11) $1 - \frac{x}{2} - \frac{x^2}{2.4} - \frac{1.3}{2.4.6}x^3 - \frac{1.3.5x^4}{2.4.6.8} \cdots\cdots \infty = \sqrt{1-x}$ (B. Series, $n = +\frac{1}{2}$)

To Prove $1 - e^{iA} = e^{\frac{iA}{2}}\left(-2i\sin\frac{A}{2}\right)$

LHS $1 - e^{iA} = e^{\frac{iA}{2}}\left(e^{\frac{-iA}{2}} - e^{\frac{iA}{2}}\right) = (2i)e^{\frac{iA}{2}}\left(\dfrac{e^{\frac{-iA}{2}} - e^{\frac{iA}{2}}}{2i}\right)$

$\dfrac{e^{\frac{iA}{2}} - e^{\frac{-iA}{2}}}{2i} = \sin\frac{A}{2}$

x & ÷ by $2i$, Take − common $= -2i\, e^{\frac{iA}{2}}\left(\dfrac{e^{\frac{iA}{2}} - e^{\frac{-iA}{2}}}{2i}\right)$

$= -2i\, e^{\frac{iA}{2}} \sin\frac{A}{2}$

To Prove $e^{\theta} = \sinh\theta + \cosh\theta$

RHS $= \dfrac{e^{\theta} - e^{-\theta}}{2} + \dfrac{e^{\theta} + e^{-\theta}}{2} = \dfrac{e^{\theta} \cancel{- e^{-\theta}} + e^{\theta} \cancel{+ e^{-\theta}}}{2} = \dfrac{2e^{\theta}}{2} = e^{\theta} = LHS.$

Ex 1.5

For each of problem 1–5, evaluate the indicated sum.

(i) $\sin A + \sin 2A + \sin 3A + \ldots \sin nA$.

(ii) $\cos A + \cos 2A + \cos 3A + \ldots \cos nA$.

Sol: Let $S = \sin A + \sin 2A + \sin 3A + \ldots + \sin nA$

& $C = \cos A + \cos 2A + \cos 3A + \ldots + \cos nA$

$$\Rightarrow C + iS = (\cos A + i\sin A) + (\cos 2A + i\sin 2A) + \ldots + (\cos nA + i\sin nA)$$

$$= e^{iA} + e^{2iA} + e^{3iA} + \ldots e^{niA}$$

Geometric Series with $a = e^{iA}$, $n = n$, $r = e^{iA}$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\Rightarrow C + iS = e^{iA}\frac{(e^{niA} - 1)}{e^{iA} - 1}$$

Taking $e^{ni\frac{A}{2}}$ common from N

Taking $e^{i\frac{A}{2}}$ common from D

$$= e^{iA}\left\{\frac{e^{ni\frac{A}{2}}\left(e^{\frac{niA}{2}} - e^{-\frac{niA}{2}}\right)}{e^{i\frac{A}{2}}\left(e^{i\frac{A}{2}} - e^{-i\frac{A}{2}}\right)}\right\}$$

$$= e^{iA} \cdot e^{ni\frac{A}{2}} \cdot e^{-i\frac{A}{2}} \frac{\left(e^{\frac{niA}{2}} - e^{-\frac{niA}{2}}\right)}{\left(e^{i\frac{A}{2}} - e^{-i\frac{A}{2}}\right)}$$

$$= e^{\frac{2iA + niA - iA}{2}} \frac{\frac{\left(e^{\frac{niA}{2}} - e^{-\frac{niA}{2}}\right)}{2i}}{\frac{\left(e^{i\frac{A}{2}} - e^{-i\frac{A}{2}}\right)}{2i}}$$

÷ N & D by $2i$

$$= e^{iA\left(\frac{1+n}{2}\right)} \cdot \frac{\sin \frac{nA}{2}}{\sin \frac{A}{2}}$$

$$= \left[\cos\left(\frac{1+n}{2}\right)A + i\sin\left(\frac{1+n}{2}\right)A\right]\frac{\sin\frac{nA}{2}}{\sin\frac{A}{2}}$$

Comparing Real & Imaginary Parts

$$C = \cos\left(\frac{1+n}{2}\right)A \cdot \frac{\sin\frac{nA}{2}}{\sin\frac{A}{2}}$$

& $S = \sin\left(\frac{1+n}{2}\right)A \cdot \dfrac{\sin\frac{nA}{2}}{\sin\frac{A}{2}}$

$$\cos A + \cos 2A + \cos 3A \ldots \cos nA = \cos\left(\frac{1+n}{2}\right)A \cdot \frac{\sin\frac{nA}{2}}{\sin\frac{A}{2}}$$

& $\sin A + \sin 2A + \ldots \sin nA = \sin\left(\frac{n+1}{2}\right)A \cdot \dfrac{\sin\frac{nA}{2}}{\sin\frac{A}{2}}$

② Let $C = \cos\theta + \cos 3\theta + \cos 5\theta + \cdots \cos(2n-1)\theta$

Sol & $S = \sin\theta + \sin 3\theta + \sin 5\theta + \cdots \sin(2n-1)\theta$

$\Rightarrow C+iS = (\cos\theta + i\sin\theta) + (\cos 3\theta + i\sin 3\theta) + \cdots\cdots (\cos(2n-1)\theta + i\sin(2n-1)\theta)$

$$= e^{i\theta} + e^{3i\theta} + e^{5i\theta} + \cdots e^{(2n-1)i\theta}$$

Geometric Series
$a = e^{i\theta}, n = n, r = e^{2i\theta}$
$S_n = \dfrac{a(r^n - 1)}{r-1}$

$$\Rightarrow C+iS = e^{i\theta}\frac{(e^{2ni\theta} - 1)}{(e^{2i\theta} - 1)}$$

$$= \cancel{e^{i\theta}}\left(\frac{e^{ni\theta}(e^{ni\theta} - e^{-ni\theta})}{\cancel{e^{i\theta}}(e^{i\theta} - e^{-i\theta})}\right)$$

$$= \frac{e^{ni\theta}\left(\dfrac{e^{ni\theta} - e^{-ni\theta}}{2i}\right)}{\left(\dfrac{e^{i\theta} - e^{-i\theta}}{2i}\right)}$$

$\div$ N & D by $2i$

$$\Rightarrow C+iS = (\cos n\theta + i\sin n\theta)\left(\frac{\sin n\theta}{\sin\theta}\right)$$

Comparing Real part only

$$C = \cos n\theta \cdot \frac{\sin n\theta}{\sin\theta}$$

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cdots \cos(2n-1)\theta = \cos n\theta \cdot \frac{\sin n\theta}{\sin\theta}$$

$$= \frac{2}{2}\cos n\theta \frac{\sin n\theta}{\sin\theta}$$

$$= \frac{\sin 2n\theta}{2\sin\theta}$$

(2) $1+x\cos\theta+x^2\cos 2\theta+\dots x^n\cos n\theta$

Sol: Let $C = 1+x\cos\theta+x^2\cos2\theta+\dots x^n\cos n\theta$

& $S = x\sin\theta+x^2\sin2\theta+\dots x^n\sin n\theta$

$$C+iS = 1+x(\cos\theta+i\sin\theta)+x^2(\cos2\theta+i\sin2\theta)+\dots x^n(\cos n\theta+i\sin n\theta)$$

$$= 1+xe^{i\theta}+x^2e^{i2\theta}+x^3e^{i3\theta}+\dots+x^ne^{in\theta}$$

G. Series
$a=1,\ r=xe^{i\theta}$
$n=n+1$
$S_n = \dfrac{a(r^n-1)}{r-1}$

$$C+iS = 1\left[\frac{(xe^{i\theta})^{n+1}-1}{xe^{i\theta}-1}\right] = \frac{x^{n+1}e^{(n+1)i\theta}-1}{xe^{i\theta}-1}$$

$$= \frac{x^{n+1}\{\cos(n+1)\theta+i\sin(n+1)\theta\}-1}{x(\cos\theta+i\sin\theta)-1}$$

$$= \frac{(x^{n+1}\cos(n+1)\theta-1)+i(x^{n+1}\sin(n+1)\theta)}{(x\cos\theta-1)+i(x\sin\theta)}$$

$$= \frac{(x^{n+1}\cos(n+1)\theta-1)+i(x^{n+1}\sin(n+1)\theta)}{(x\cos\theta-1)+i(x\sin\theta)}\cdot\frac{(x\cos\theta-1)-i(x\sin\theta)}{(x\cos\theta-1)-i(x\sin\theta)}$$

$$C+iS = \frac{\left[(x^{n+1}\cos(n+1)\theta-1)(x\cos\theta-1)+(x^{n+1}\sin(n+1)\theta)(x\sin\theta)\right]+i\left[(x^{n+1}\sin(n+1)\theta)(x\cos\theta-1)-(x^{n+1}\cos(n+1)\theta-1)(x\sin\theta)\right]}{(x\cos\theta-1)^2+(x\sin\theta)^2}$$

Comparing Real part only

$$C = \frac{x^{n+2}\cos(n+1)\theta\cos\theta-x^{n+1}\cos(n+1)\theta-x\cos\theta+1+x^{n+2}\sin(n+1)\theta\sin\theta}{x^2\cos^2\theta+1-2x\cos\theta+x^2\sin^2\theta}$$

$$= \frac{x^{n+2}\left[\cos(n+1)\theta\cos\theta+\sin(n+1)\theta\sin\theta\right]-x^{n+1}\cos(n+1)\theta-x\cos\theta+1}{x^2(\cos^2\theta+\sin^2\theta)-2x\cos\theta+1}$$

$$C = \frac{x^{n+2}\cos[(n+1)\theta-\theta]-x^{n+1}\cos(n+1)\theta-x\cos\theta+1}{x^2-2x\cos\theta+1}$$

$$C = \frac{x^{n+2}\cos n\theta-x^{n+1}\cos(n+1)\theta-x\cos\theta+1}{x^2-2x\cos\theta+1}$$ Ans

(4) $S = 3\sin\alpha + 5\sin 2\alpha + 7\sin 3\alpha + \cdots (2n+1)\sin n\alpha$

Sol Let $S = 3\sin\alpha + 5\sin 2\alpha + 7\sin 3\alpha + \cdots (2n+1)\sin n\alpha$

& $C = 3\cos\alpha + 5\cos 2\alpha + 7\sin 3\alpha + \cdots (2n+1)\cos n\alpha$

$C+iS = 3(\cos\alpha + i\sin\alpha) + 5(\cos 2\alpha + i\sin 2\alpha) + \cdots (2n+1)(\cos n\alpha + i\sin n\alpha)$

$C+iS = 3e^{i\alpha} + 5e^{i2\alpha} + 7e^{i3\alpha} + \cdots (2n+1)e^{in\alpha}$ —— (1) (Not G. Series)

Multiply eq (1) by $e^{i\alpha}$ & subtract

$(C+iS)e^{i\alpha} = \quad + 3e^{i2\alpha} + 5e^{i3\alpha} + 7e^{i4\alpha} \cdots (2n-1)e^{in\alpha} + (2n+1)e^{i(n+1)\alpha}$ —— (2)

$(C+iS)(1-e^{i\alpha}) = 3e^{i\alpha} + 2e^{i2\alpha} + 2e^{i3\alpha} + \cdots + 2e^{in\alpha} - (2n+1)e^{i(n+1)\alpha}$

$= (e^{i\alpha} + 2e^{i\alpha}) + 2e^{i2\alpha} + 2e^{i3\alpha} + \cdots 2e^{in\alpha} - (2n+1)e^{i(n+1)\alpha}$

$= e^{i\alpha} + 2(e^{i\alpha} + e^{i2\alpha} + e^{i3\alpha} + \cdots e^{in\alpha}) - (2n+1)e^{i(n+1)\alpha}$

$= e^{i\alpha} + 2\left[\dfrac{e^{i\alpha}(e^{ni\alpha} - 1)}{(e^{i\alpha} - 1)}\right] - (2n+1)e^{i(n+1)\alpha}$ G. Series $a = e^{i\alpha}$, $r = e^{i\alpha}$, $n = n$

LCM $= \dfrac{e^{i\alpha}(e^{i\alpha}-1) + 2e^{i\alpha}(e^{ni\alpha}-1) - (2n+1)e^{i(n+1)\alpha}\cdot(e^{i\alpha}-1)}{(e^{i\alpha}-1)}$

$(C+iS)(1-e^{i\alpha})(e^{i\alpha}-1) = e^{i\alpha}\left[e^{i\alpha} - 1 + 2e^{ni\alpha} - 2 - (2n+1)e^{i(n+1)\alpha} + (2n+1)e^{ni\alpha}\right]$ ✓

$-(C+iS)(e^{i\alpha}-1)^2 = e^{i\alpha}\left[e^{i\alpha} - 3 + (2+2n+1)e^{ni\alpha} - (2n+1)e^{i(n+1)\alpha}\right]$

$-(C+iS)\left[-4e^{i\alpha}\sin^2\frac{\alpha}{2}\right] = e^{i\alpha}\left[e^{i\alpha} - 3 + (2n+3)e^{ni\alpha} - (2n+1)e^{i(n+1)\alpha}\right]$

$(C+iS) = \dfrac{\cancel{e^{i\alpha}}\left(e^{i\alpha} - 3 + (2n+3)e^{ni\alpha} - (2n+1)e^{i(n+1)\alpha}\right)}{4\cancel{e^{i\alpha}}\sin^2\frac{\alpha}{2}}$

$(e^{i\alpha} - 1)^2$

$= \left[e^{\frac{i\alpha}{2}}\left(e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}}\right)\right]^2$

$= \left[e^{\frac{i\alpha}{2}}\left(\dfrac{e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}}}{2i}\right)2i\right]^2$

$= e^{\left(\frac{i\alpha}{2}\right)^2}\sin^2\frac{\alpha}{2}(-4)$

$= -4e^{i\alpha}\sin^2\frac{\alpha}{2}$

$= \dfrac{\cos\alpha + i\sin\alpha - 3 + (2n+3)(\cos n\alpha + i\sin n\alpha) - (2n+1)(\cos(n+1)\alpha + i\sin(n+1)\alpha)}{4\left(\frac{1-\cos\alpha}{2}\right)}$

Comparing Imaginary Part

$S = \dfrac{\sin\alpha + (2n+3)\sin n\alpha - (2n+1)\sin(n+1)\alpha}{2(1-\cos\alpha)}$

(5) $\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta + \cdots \cos^2 n\theta$

$$= \frac{1+\cos 2\theta}{2} + \frac{1+\cos 4\theta}{2} + \frac{1+\cos 6\theta}{2} + \cdots \frac{1+\cos 2n\theta}{2}$$

$$= \left(\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \cdots n\text{ terms}\right) + \frac{\cos 2\theta}{2} + \frac{\cos 4\theta}{2} + \frac{\cos 6\theta}{2} \cdots \frac{\cos 2n\theta}{2}$$

$$S_n = \frac{n}{2} + \frac{1}{2}(\cos 2\theta + \cos 4\theta + \cos 6\theta + \cdots \cos 2n\theta) \quad \text{—(1)}$$

Let $C = \cos 2\theta + \cos 4\theta + \cos 6\theta + \cdots \cos 2n\theta$

$S = \sin 2\theta + \sin 4\theta + \sin 6\theta + \cdots \sin 2n\theta$

$$C + iS = e^{2i\theta} + e^{4i\theta} + e^{6i\theta} + \cdots e^{2ni\theta}$$

Geometric Series $a = e^{2i\theta}$, $r = e^{2i\theta}$, $n = n$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$$C + iS = \frac{e^{2i\theta}\left(e^{i2\theta n} - 1\right)}{e^{2i\theta} - 1}$$

$$= \frac{e^{2i\theta} \cdot e^{ni\theta}\left(e^{ni\theta} - e^{-ni\theta}\right)}{e^{i\theta}\left(e^{i\theta} - e^{-i\theta}\right)}$$

$$= e^{i\theta} \cdot e^{ni\theta} \frac{\left(e^{ni\theta} - e^{-ni\theta}\right)}{\left(e^{i\theta} - e^{-i\theta}\right)}$$

$$= e^{(n+1)i\theta} \frac{\left(\frac{e^{ni\theta} - e^{-ni\theta}}{2i}\right)}{\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)}$$

$$= e^{(n+1)i\theta} \cdot \frac{\sin n\theta}{\sin\theta}$$

$$C + iS = \left(\cos(n+1)\theta + i\sin(n+1)\theta\right)\frac{\sin n\theta}{\sin\theta}$$

Comparing Real Part

$$C = \cos(n+1)\theta \, \frac{\sin n\theta}{\sin\theta}$$

Hence from (1) $S_n = \frac{n}{2} + \frac{1}{2}\,\frac{\sin n\theta}{\sin\theta}\cos(n+1)\theta$

(6) Find sum of Infinite series.

Let $S = \sin\theta + \frac{1}{2}\sin 3\theta + \frac{1\cdot 3}{2\cdot 4}\sin 5\theta + \cdots \infty$

$C = \cos\theta + \frac{1}{2}\cos 3\theta + \frac{1\cdot 3}{2\cdot 4}\cos 5\theta + \cdots \infty$

$$C+iS = e^{i\theta} + \frac{1}{2}e^{3i\theta} + \frac{1\cdot 3}{2\cdot 4}e^{5i\theta} + \cdots \infty$$

$$= e^{i\theta}\left(1 + \frac{1}{2}e^{2i\theta} + \frac{1\cdot 3}{2\cdot 4}e^{4i\theta} + \cdots \infty\right)$$

$$= e^{i\theta}\left(1 - e^{2i\theta}\right)^{-\frac{1}{2}} \qquad \because (1-x)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots$$

$$= e^{i\theta}\left(e^{2i\theta}\right)^{-\frac{1}{2}}\left(e^{-2i\theta} - 1\right)^{-\frac{1}{2}}$$

$$= e^{i\theta}e^{-i\theta}\left(\cos(-2\theta) + i\sin(-2\theta) - 1\right)^{-\frac{1}{2}}$$

$$= e^{0}\left(\cos 2\theta - i\sin 2\theta - 1\right)^{-\frac{1}{2}}$$

$$= \left[(\cos 2\theta - 1) - i\sin 2\theta\right]^{-\frac{1}{2}} \qquad \cos 2\theta = 1 - 2\sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \left(-2\sin^2\theta - i2\sin\theta\cos\theta\right)^{-\frac{1}{2}}$$

$$= \left\{2\sin\theta(-\sin\theta - i\cos\theta)\right\}^{-\frac{1}{2}}$$

$$= (2\sin\theta)^{-\frac{1}{2}}\left(-\sin\theta - i\cos\theta\right)^{-\frac{1}{2}}$$

$$= (2\sin\theta)^{-\frac{1}{2}}\left[\cos\left(\frac{\pi}{2}+\theta\right) - i\sin\left(\frac{\pi}{2}+\theta\right)\right]^{-\frac{1}{2}} \qquad \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta, \quad \sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$$

$$= (2\sin\theta)^{-\frac{1}{2}}\left[\cos\left(-\frac{\pi}{4}-\frac{\theta}{2}\right) - i\sin\left(-\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

$$C+iS = (2\sin\theta)^{-\frac{1}{2}}\left[\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$$

Comparing Imaginary parts

$$S = (2\sin\theta)^{-\frac{1}{2}}\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$S = \frac{\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}{\sqrt{2\sin\theta}} \quad \text{Ans.}$$

(7) $\sinh\theta + \frac{\sinh 2\theta}{\lfloor 2} + \frac{\sinh 3\theta}{\lfloor 3} + \cdots \infty$

Sol $= \left(\frac{e^{\theta} - e^{-\theta}}{2}\right) + \left(\frac{e^{2\theta} - e^{-2\theta}}{2\lfloor 2}\right) + \left(\frac{e^{3\theta} - e^{-3\theta}}{2\lfloor 3}\right) + \cdots \infty$

$= \frac{1}{2}\left[e^{\theta} - e^{-\theta} + \frac{e^{2\theta} - e^{-2\theta}}{\lfloor 2} + \frac{e^{3\theta} - e^{-3\theta}}{\lfloor 3} + \cdots \infty\right]$

$= \frac{1}{2}\left[e^{\theta} + \frac{e^{2\theta}}{\lfloor 2} + \frac{e^{3\theta}}{\lfloor 3} + \cdots \infty\right] - \frac{1}{2}\left[e^{-\theta} + \frac{e^{-2\theta}}{\lfloor 2} + \frac{e^{-3\theta}}{\lfloor 3} + \cdots \infty\right]$

Add & Subtract 1

$\Rightarrow \frac{1}{2}\left[1 + e^{\theta} + \frac{e^{2\theta}}{\lfloor 2} + \frac{e^{3\theta}}{\lfloor 3} + \cdots \infty\right] - \frac{1}{2}\left[1 + e^{-\theta} + \frac{e^{-2\theta}}{\lfloor 2} + \frac{e^{-3\theta}}{\lfloor 3} + \cdots \infty\right]$

$\because e^{x} = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots \infty$

$= \frac{1}{2}\left(e^{e^{\theta}}\right) - \frac{1}{2}\left(e^{e^{-\theta}}\right)$

$= \frac{1}{2}\left[e^{e^{\theta}} - e^{e^{-\theta}}\right]$

$*$ $= \frac{1}{2}\left[e^{\cosh\theta + \sinh\theta} - e^{\cosh\theta - \sinh\theta}\right]$

$= \frac{e^{\cosh\theta}}{2}\left(e^{\sinh\theta} - e^{-\sinh\theta}\right)$

$= e^{\cosh\theta}\left(\frac{e^{\sinh\theta} - e^{-\sinh\theta}}{2}\right)$

$= e^{\cosh\theta}\sinh(\sinh\theta)$ Ans

$*$ $\cosh\theta + \sinh\theta = e^{\theta}$

LHS $= \frac{e^{\theta} + e^{-\theta}}{2} + \frac{e^{\theta} - e^{-\theta}}{2} = \frac{e^{\theta} + \cancel{e^{-\theta}} + e^{\theta} - \cancel{e^{-\theta}}}{2}$

$= \frac{2e^{\theta}}{2} = e^{\theta}$ RHS

$=$

Q13 $C = 1 + c\cos\theta + \frac{c^2}{\lfloor 2}\cos 2\theta + \frac{c^3}{\lfloor 3}\cos 3\theta + \cdots$

$S = c\sin\theta + \frac{c^2}{\lfloor 2}\sin 2\theta + \frac{c^3}{\lfloor 3}\sin 3\theta + \cdots$

$C + iS = 1 + c(\cos\theta + i\sin\theta) + \frac{c^2}{\lfloor 2}(\cos 2\theta + i\sin 2\theta) + \cdots$

$= 1 + ce^{i\theta} + \frac{c^2}{\lfloor 2}e^{i2\theta} + \frac{c^3}{\lfloor 3}e^{3i\theta} + \cdots$

$= 1 + ce^{i\theta} + \frac{(ce^{i\theta})^2}{\lfloor 2} + \frac{(ce^{i\theta})^3}{\lfloor 3} + \cdots$

$= e^{ce^{i\theta}} \quad \because e^{x} = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots \infty$

$= e^{c(\cos\theta + i\sin\theta)}$

$= e^{c\cos\theta} \cdot e^{ic\sin\theta}$

$C + iS = e^{c\cos\theta}\left(\cos(c\sin\theta) + i\sin(c\sin\theta)\right)$

Comparing Real Part

$C = e^{c\cos\theta}\cos(c\sin\theta)$ Ans.

Q8 $S = \sin\alpha \cdot \sin\alpha + \sin^2\alpha \cdot \sin 2\alpha + \sin^3\alpha \cdot \sin 3\alpha + \cdots \infty$

$C = \sin\alpha \cdot \cos\alpha + \sin^2\alpha \cdot \cos 2\alpha + \sin^3\alpha \cdot \cos 3\alpha + \cdots \infty$

$$C + iS = \sin\alpha(\cos\alpha + i\sin\alpha) + \sin^2\alpha(\cos 2\alpha + i\sin 2\alpha) + \cdots \infty$$

$$= \sin\alpha\, e^{i\alpha} + \sin^2\alpha\, e^{2i\alpha} + \sin^3\alpha\, e^{3i\alpha} + \cdots \infty$$ Infinite Geometric Series

$a = \sin\alpha\, e^{i\alpha}$

$r = \sin\alpha\, e^{i\alpha}$

$S_\infty = \dfrac{a}{1-r}$

$$C + iS = \frac{a}{1-r} = \frac{\sin\alpha\, e^{i\alpha}}{1 - \sin\alpha\, e^{i\alpha}}$$

$$= \frac{\sin\alpha\, \cancel{e^{i\alpha}}}{\cancel{e^{i\alpha}}(e^{-i\alpha} - \sin\alpha)}$$

$$= \frac{\sin\alpha}{(\cos\alpha - i\sin\alpha) - \sin\alpha}$$

$$= \frac{\sin\alpha}{(\cos\alpha - \sin\alpha) - i\sin\alpha}$$

$$= \frac{\sin\alpha}{(\cos\alpha - \sin\alpha) - i\sin\alpha} \cdot \frac{[(\cos\alpha - \sin\alpha) + i\sin\alpha]}{[(\cos\alpha - \sin\alpha) + i\sin\alpha]}$$

$$= \frac{\sin\alpha\,[(\cos\alpha - \sin\alpha) + i\sin\alpha]}{(\cos\alpha - \sin\alpha)^2 + \sin^2\alpha}$$

$$= \frac{\sin\alpha(\cos\alpha - \sin\alpha) + i\sin^2\alpha}{\cos^2\alpha + \sin^2\alpha - 2\sin\alpha\cos\alpha + \sin^2\alpha}$$

$$C + iS = \frac{\sin\alpha(\cos\alpha - \sin\alpha) + i\sin^2\alpha}{1 - \sin 2\alpha + \sin^2\alpha}$$

Comparing imaginary parts

$$S = \frac{\sin^2\alpha}{1 - \sin 2\alpha + \sin^2\alpha}$$ Ans.

1.5-10

Q(9) $C = 1-\frac{1}{2}\cos\theta + \frac{1\cdot3}{2\cdot4}\cos2\theta - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cos3\theta + \cdots\infty$

$S = -\frac{1}{2}\sin\theta + \frac{1\cdot3}{2\cdot4}\sin2\theta - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\sin3\theta + \cdots\infty$

$C+iS = 1-\frac{1}{2}(\cos\theta+i\sin\theta) + \frac{1\cdot3}{2\cdot4}(\cos2\theta+i\sin2\theta) - \cdots\infty$

$= 1-\frac{1}{2}e^{i\theta} + \frac{1\cdot3}{2\cdot4}e^{2i\theta} - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}e^{3i\theta} + \cdots\infty$

$= (1+e^{i\theta})^{-\frac{1}{2}}$ $\quad \because (1+x)^{-\frac{1}{2}} = 1-\frac{1}{2}x + \frac{1\cdot3}{2\cdot4}x^2 - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^3 + \cdots\infty$ B.Series

$= (1+\cos\theta+i\sin\theta)^{-\frac{1}{2}}$

$= \left(2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}$

$= \left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right]^{-\frac{1}{2}}$

DeMoivres

$= \left(2\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}\left(\cos\frac{\theta}{4} - i\sin\frac{\theta}{4}\right)$

$C+iS = \dfrac{\cos\frac{\theta}{4} - i\sin\frac{\theta}{4}}{\sqrt{2\cos\frac{\theta}{2}}}$

Comparing Real Part

$C = \dfrac{\cos\frac{\theta}{4}}{\sqrt{2\cos\frac{\theta}{2}}}$ Ans

x————x

Q(11) $C = 1+\frac{1}{2}\cos\theta + \frac{1\cdot3}{2\cdot4}\cos2\theta + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cos3\theta + \cdots\infty$

$S = \frac{1}{2}\sin\theta + \frac{1\cdot3}{2\cdot4}\sin2\theta + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\sin3\theta + \cdots\infty$

$C+iS \quad 1+\frac{1}{2}e^{i\theta} + \frac{1\cdot3}{2\cdot4}e^{2i\theta} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}e^{3i\theta} + \cdots\infty$

$= (1-e^{i\theta})^{-\frac{1}{2}}$ $\quad \because (1-x)^{-\frac{1}{2}} = 1+\frac{1}{2}x + \frac{1\cdot3}{2\cdot4}x^2 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^3 + \cdots$ B.Series

$= (1-\cos\theta - i\sin\theta)^{-\frac{1}{2}}$

$= \left(2\sin^2\frac{\theta}{2} - i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}$

$= \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}$

$= \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left[\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]^{-\frac{1}{2}}$

$= \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left(\cos\left(\frac{\pi-\theta}{4}\right) + i\sin\left(\frac{\pi-\theta}{4}\right)\right)$

DeMoivres Th.

Comparing Real Part

$C = \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left(\cos\left(\frac{\pi-\theta}{4}\right)\right)$

$C = \dfrac{\cos\left(\frac{\pi-\theta}{4}\right)}{\sqrt{2\sin\frac{\theta}{2}}}$ Ans

x————x

(10) $S = n\sin\theta + \dfrac{n(n+1)}{|\underline{2}}\sin 2\theta + \dfrac{n(n+1)(n+2)}{|\underline{3}}\sin 3\theta + \cdots\cdots\infty$

$C = 1 + n\cos\theta + \dfrac{n(n+1)}{|\underline{2}}\cos 2\theta + \dfrac{n(n+1)(n+2)}{|\underline{3}}\cos 3\theta + \cdots\cdots\infty$

$$C+iS = 1 + n(\cos\theta + i\sin\theta) + \frac{n(n+1)}{|\underline{2}}(\cos 2\theta + i\sin 2\theta) + \frac{n(n+1)(n+2)}{|\underline{3}}(\cos 3\theta + i\sin 3\theta) + \cdots$$

$$= 1 + n e^{i\theta} + \frac{n(n+1)}{|\underline{2}} e^{2i\theta} + \frac{n(n+1)(n+2)}{|\underline{3}} e^{3i\theta} + \cdots\cdots\infty$$

$$= 1 + (-n)(-e^{i\theta}) + \frac{(-n)(-n-1)}{|\underline{2}}(-e^{i\theta})^2 + \frac{(-n)(-n-1)(-n-2)}{|\underline{3}}(-e^{i\theta})^3 + \cdots\cdots\infty$$

$$= \left(1 - e^{i\theta}\right)^{-n} \qquad \because \text{B. Series } (1-x)^{-n} = 1 + nx + \frac{n(n-1)}{|\underline{2}}x^2 + \cdots\cdots\infty$$

$$= (1 - \cos\theta - i\sin\theta)^{-n}$$

$$= \left(2\sin^2\frac{\theta}{2} - i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-n} \qquad \cos\theta = 1 - 2\sin^2\frac{\theta}{2},\quad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{-n}\left[\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right]^{-n} \qquad \cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = \sin\frac{\theta}{2},\quad \sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = \cos\frac{\theta}{2}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{-n}\left[\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]^{-n}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{-n}\left[\cos\left(-n\left(\frac{\pi-\theta}{2}\right)\right) - i\sin\left(-n\left(\frac{\pi-\theta}{2}\right)\right)\right] \qquad \text{Demoivres Th.}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{-n}\left[\cos\left(\frac{n\pi - n\theta}{2}\right) + i\sin\left(\frac{n\pi - n\theta}{2}\right)\right]$$

Comparing Imaginary parts

$$S = \left(2\sin\frac{\theta}{2}\right)^{-n}\sin\left(\frac{n\pi - n\theta}{2}\right)$$

$$S = \frac{\sin\frac{n}{2}(\pi-\theta)}{\left(2\sin\frac{\theta}{2}\right)^n} \quad \text{Ans.}$$

1.5-12

(13)

$$C = \cos\alpha - \frac{\cos(\alpha+2\beta)}{\lfloor 3} + \frac{\cos(\alpha+4\beta)}{\lfloor 5} + \cdots\cdots \infty$$

$$S = \sin\alpha - \frac{\sin(\alpha+2\beta)}{\lfloor 3} + \frac{\sin(\alpha+4\beta)}{\lfloor 5} + \cdots\cdots \infty$$

$$C + iS = e^{i\alpha} - \frac{e^{i(\alpha+2\beta)}}{\lfloor 3} + \frac{e^{i(\alpha+4\beta)}}{\lfloor 5} + \cdots\cdots \infty$$

$$= e^{i\alpha}\left(1 - \frac{e^{i2\beta}}{\lfloor 3} + \frac{e^{i4\beta}}{\lfloor 5} - \cdots\cdots \infty\right)$$

$$= e^{i\alpha} \cdot \frac{e^{i\beta}}{e^{i\beta}}\left(1 - \frac{e^{i2\beta}}{\lfloor 3} + \frac{e^{i4\beta}}{\lfloor 5} - \cdots\cdots \infty\right)$$

$$= \frac{e^{i\alpha}}{e^{i\beta}}\left(e^{i\beta} - \frac{e^{i3\beta}}{\lfloor 3} + \frac{e^{i5\beta}}{\lfloor 5} - \cdots\cdots \infty\right)$$

$$= e^{i(\alpha-\beta)}\left(\sin e^{i\beta}\right) \qquad \because \sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots\cdots \infty$$

$$= \left(\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right) \sin e^{i\beta}$$

$$= \left(\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right) \sin(\cos\beta + i\sin\beta)$$

$$= \left(\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right)\left[\sin(\cos\beta)\cos(i\sin\beta) + \cos(\cos\beta)\sin(i\sin\beta)\right]$$

$$= \left[\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right]\left[\sin(\cos\beta)\cos(i\sin\beta) + \cos(\cos\beta)\sin(i\sin\beta)\right]$$

$$= \left[\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right]\left[\sin(\cos\beta)\cosh(\sin\beta) + i\cos(\cos\beta)\sinh(\sin\beta)\right]$$

$$C + iS = \left[\cos(\alpha-\beta)\sin(\cos\beta)\cosh(\sin\beta) - \sin(\alpha-\beta)\cos(\cos\beta)\sinh(\sin\beta)\right]$$
$$+ i\left[\sin(\alpha-\beta)\sin\cos\beta\cosh(\sin\beta) + \cos(\alpha-\beta)\cos(\cos\beta)\sinh(\sin\beta)\right]$$

Comparing Real Part

$$C = \cos(\alpha-\beta)\sin(\cos\beta)\cosh(\sin\beta) - \sin(\alpha-\beta)\cos(\cos\beta)\sinh(\sin\beta)$$

Ans.

Q14 $C = c\cos\theta + \frac{c^2}{2}\cos 2\theta + \frac{c^3}{3}\cos 3\theta + \cdots\cdots$

$S = c\sin\theta + \frac{c^2}{2}\sin 2\theta + \frac{c^3}{3}\sin 3\theta + \cdots\cdots$

$$C+iS = c(\cos\theta + i\sin\theta) + \frac{c^2}{2}(\cos 2\theta + i\sin 2\theta) + \frac{c^3}{3}(\cos 3\theta + i\sin 3\theta) + \cdots$$

$$= c\,e^{i\theta} + \frac{c^2}{2}e^{i2\theta} + \frac{c^3}{3}e^{i3\theta} + \cdots\cdots$$

$$= ce^{i\theta} + \frac{(ce^{i\theta})^2}{2} + \frac{(ce^{i\theta})^3}{3} + \cdots\cdots$$

$$= -\ln(1-ce^{i\theta})$$

$\because \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\cdots$

$\ln(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right.$

$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\cdots$

$$= -\mathrm{Log}(1-ce^{i\theta})$$

$$= -\log(\overline{1-c\cos\theta - ic\sin\theta})$$

$\because \mathrm{Log}(x+iy) = \ln|x+iy| + i\,\mathrm{Arg}\,z$

$$= -\left[\ln|\overline{1-c\cos\theta} - ic\sin\theta| + i\arg(\overline{1-c\cos\theta} - ic\sin\theta)\right.$$

$$C+iS = -\left[\ln\sqrt{(1-c\cos\theta)^2 + c^2\sin^2\theta} + i\tan^{-1}\left(\frac{-c\sin\theta}{1-c\cos\theta}\right)\right]$$

Comparing Real Part

$$C = -\left[\ln\sqrt{(1-c\cos\theta)^2 + c^2\sin^2\theta}\right]$$

$$= -\frac{1}{2}\ln\{(1-c\cos\theta)^2 + c^2\sin^2\theta\}$$

$$= -\frac{1}{2}\ln(1 + c^2\cos^2\theta - 2c\cos\theta + c^2\sin^2\theta)$$

$$= -\frac{1}{2}\ln(1 + c^2(\cos^2\theta + \sin^2\theta) - 2c\cos\theta)$$

$$= -\frac{1}{2}\ln(1 + c^2 - 2c\cos\theta) \quad \text{Ans}$$

(1.5-14)

(15) $S = \sin\theta - \frac{1}{2}\sin 3\theta + \frac{1}{3}\sin 5\theta + \cdots$

$C = \cos\theta - \frac{1}{2}\cos 3\theta + \frac{1}{3}\cos 5\theta + \cdots$

$C+iS = e^{i\theta} - \frac{1}{2}e^{i3\theta} + \frac{1}{3}e^{i5\theta}, \cdots$

$x+iy \; e^{i\theta}$. $= \frac{e^{i\theta}}{e^{i\theta}}\left(e^{i\theta} - \frac{1}{2}e^{i3\theta} + \frac{1}{3}e^{i5\theta} \cdots\right)$

$= \frac{1}{e^{i\theta}}\left(e^{2i\theta} - \frac{1}{2}e^{4i\theta} + \frac{1}{3}e^{6i\theta} \cdots\right)$

$= e^{-i\theta}\left[e^{2i\theta} - \frac{1}{2}(e^{2i\theta})^2 + \frac{1}{3}(e^{2i\theta})^3 \cdots\right]$

$= e^{-i\theta}\left[\operatorname{Log}(1+e^{2i\theta})\right]$ $\quad \because \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \infty$

$= e^{-i\theta}\left[\operatorname{Log}(1+\cos 2\theta + i\sin 2\theta)\right]$

$= e^{-i\theta}\left[\ln\sqrt{(1+\cos 2\theta)^2 + (\sin 2\theta)^2} + i\tan^{-1}\frac{\sin 2\theta}{1+\cos 2\theta}\right.$

$= e^{-i\theta}\left[\ln\sqrt{1+\cos^2 2\theta + 2\cos 2\theta + \sin^2 2\theta} + i\tan^{-1}\left(\frac{\cancel{2}\sin\theta\cancel{\cos\theta}}{\cancel{2}\cos^{\cancel{2}}\theta}\right)\right.$

$= e^{-i\theta}\left[\ln\sqrt{1+1+2\cos 2\theta} + i\tan^{-1}\tan\theta\right.$

$= e^{-i\theta}\left[\ln\sqrt{2(1+\cos 2\theta)} + i\theta\right]$

$= e^{-i\theta}\left[\ln\sqrt{2(2\cos^2\theta)} + i\theta\right]$

$= e^{-i\theta}\left[\ln(2\cos\theta) + i\theta\right]$

$= (\cos\theta - i\sin\theta)(\ln 2\cos\theta + i\theta)$

$= \cos\theta\ln 2\cos\theta + \theta\sin\theta + i(\theta\cos\theta - \sin\theta\ln(2\cos\theta))$

Comparing Imaginary parts

$S = \theta\cos\theta - \sin\theta\ln(2\cos\theta)$ Ans