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① Ch#03

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M.M MATRICES Matrics

Matrix:

Any rectangular arrangement of real or complex nos, subject to certain rules of operations is called a 'Matrix'. If there are 'm' rows & 'n' columns then order of matrix is $m\times n$

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$\longrightarrow$ Rows

$\downarrow$ Columns

Tabular Form

In abbreviated form, $A=[a_{ij}]_{m\times n}$ $\quad i=1,2,3,\dots m$, $\; j=1,2,3,\dots n$

Types Of Matrices:-

Square Matrix If $m=n$ i.e No of Rows = No of Column

e.g $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$

Rectangular Matrix If $m\neq n$

Diagonal Matrix If all entries of a square matrix are zero except main diagonal entries.

e.g $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ $\quad a_{ij}=0$ when $i\neq j$, $\; a_{ij}\neq 0$ when $i=j$

Identity Matrix If all entries of a square matrix are zero except main diagonal entries which are '1'.

$I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Zero or Null Matrix If all entries of a matrix are zero. $O=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Scalar Matrix The matrix obtained by multiplying a non-zero scalar to each of its entries is called Scalar Matrix.

$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ $\quad KA=\begin{bmatrix} Ka_{11} & Ka_{12} & Ka_{13} \\ Ka_{21} & Ka_{22} & Ka_{23} \end{bmatrix}$

Upper Triangular Matrix:- A square matrix whose elements below the main diagonal are all zero.

$$A = \begin{bmatrix} 1 & 7 & 9 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 7 & 5 \end{bmatrix}$$

Upper Triangular Matrix   Lower Triang. Matrix

Lower Triangular Matrix:- A square matrix whose elements above the main diagonal are all zero.

Matrix Addition: A & B are conformable for addition if

Order of A == Order of B

$$A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix Multiplication A & B are conformable for multiplication if No of Column of A = No of Rows of B.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3} \quad B = \begin{bmatrix} 7 & 8 & 4 \\ 9 & 0 & 5 \\ 1 & 2 & 6 \end{bmatrix}_{3\times 3}$$

AB is conformable for multiplication $\because$ 2×3 same 3×3 AB

BA is not conformable for multiplication. $3\times 3 \rightarrow$ Not BA 2×3

$$AB = \begin{bmatrix} 1\times7+2\times9+3\times1 & 1\times8+2\times0+3\times2 & 1\times4+2\times5+3\times6 \\ 4\times7+5\times9+6\times1 & 4\times8+5\times0+6\times2 & 4\times4+5\times5+6\times6 \end{bmatrix}$$

$$= \begin{bmatrix} 7+18+3 & 8+0+6 & 4+10+18 \\ 28+45+6 & 32+0+12 & 16+25+36 \end{bmatrix}$$

$$AB = \begin{bmatrix} 28 & 14 & 32 \\ 79 & 44 & 77 \end{bmatrix}_{2\times 3}$$

Theorem - If the matrices A, B and C are conformable for the indicated sums and products then

i) $A(BC) = (AB)C$ Associative Law

ii) $A(B+C) = AB + AC$

iii) $(A+B)C = AC + BC$ } Distributive Law

{We prove the theorems by showing that element in the ith row & jth col of LHS = the element in the ith row & jth col of R.H.S.}

iv) $K(AB) = (KA)B = A(KB)$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \end{bmatrix}$$

3rd col $\begin{bmatrix} \sum_{\lambda=1}^{3} a_{1\lambda} b_{\lambda 3} \\ \sum_{\lambda=1}^{3} a_{2\lambda} b_{\lambda 3} \end{bmatrix}$

Proof (i) $A(BC) = (AB)C$

Let $A = [a_{ij}]_{m\times n}$ $B = [b_{ij}]_{n\times p}$

$C = [c_{ij}]_{p\times q}$

Order of $A = m\times n$

Order of $BC = n\times p \; p\times q = n\times q$

Order of $A(BC) = m\times n \; n\times q = \boxed{m\times q}$

Similarly

Order of $AB = m\times n \; n\times p = m\times p$

Order of $C = p\times q$

Order of $(AB)C = m\times p \; p\times q = \boxed{m\times q}$

Now i^{th} row of A is $\{a_{i1}, a_{i2}, a_{i3} \cdots a_{in}\}$

& j^{th} column of BC is $\begin{Bmatrix} \sum_{\lambda=1}^{p} b_{1\lambda} c_{\lambda j} \\ \sum_{\lambda=1}^{p} b_{2\lambda} c_{\lambda j} \\ \vdots \\ \sum_{\lambda=1}^{p} b_{n\lambda} c_{\lambda j} \end{Bmatrix}$

$\therefore$ the element in the ith row & jth col of $A(BC)$ is

$$a_{i1}\left(\sum_{\lambda=1}^{p} b_{1\lambda} c_{\lambda j}\right) + a_{i2}\left(\sum_{\lambda=1}^{p} b_{2\lambda} c_{\lambda j}\right) + \cdots + a_{in}\left(\sum_{\lambda=1}^{p} b_{n\lambda} c_{\lambda j}\right)$$

$$= \sum_{\mu=1}^{n} a_{i\mu}\left(\sum_{\lambda=1}^{p} b_{\mu\lambda} c_{\lambda j}\right)$$

$$= \sum_{\mu=1}^{n} \sum_{\lambda=1}^{p} a_{i\mu}\left(b_{\mu\lambda} c_{\lambda j}\right)$$

$$= \sum_{\mu=1}^{n} \sum_{\lambda=1}^{p} \left(a_{i\mu} b_{\mu\lambda}\right) c_{\lambda j}$$ $\because$ Associative Law holds in Real Nos.

$$= \sum_{\mu=1}^{n} \left(a_{i\mu} b_{\mu\lambda}\right) \sum_{\lambda=1}^{p} c_{\lambda j}$$

$$= \sum_{\lambda=1}^{p} \left(\sum_{\mu=1}^{n} a_{i\mu} b_{\mu\lambda}\right) c_{\lambda j}$$

$$= \left(\sum_{\mu=1}^{n} a_{i\mu} b_{\mu 1}\right) c_{1j} + \left(\sum_{\mu=1}^{n} a_{i\mu} b_{\mu 2}\right) c_{2j} + \cdots \left(\sum_{\mu=1}^{n} a_{i\mu} b_{\mu p}\right) c_{pj}$$

= Element in the ith row & jth col of $(AB)C$.

Hence $A(BC) = (AB)C$

$$A(B+C) = AB + AC$$

Let $A = [a_{ij}]_{m\times n}$ $B = [b_{ij}]_{n\times p}$ $C = [c_{ij}]_{n\times p}$

order of $(B+C) = n\times p$

order of $A(B+C) = m\times n \; n\times p$

$= \boxed{m\times p}$

order of $AB = m\times n \; n\times p = m\times p$

order of $AC = m\times n \; n\times p = m\times p$

order of $AB + AC = \boxed{m\times p}$

order of $A(B+C)$ = order of $AB + AC$

Now

ith row of A is $\{a_{i1}, a_{i2}, \cdots a_{in}\}$

jth column of $(B+C)$ is $\begin{Bmatrix} b_{1j} + c_{1j} \\ b_{2j} + c_{2j} \\ \vdots \\ b_{nj} + c_{nj} \end{Bmatrix}$

The element in the ith row & jth column of $A(B+C)$ is

$$a_{i1}(b_{1j}+c_{1j})+a_{i2}(b_{2j}+c_{2j})+\cdots\cdots a_{in}(b_{nj}+c_{nj})$$

$$=\sum_{\lambda=1}^{n} a_{i\lambda}(b_{\lambda j}+c_{\lambda j})$$

$$=\sum_{\lambda=1}^{n}\left[a_{i\lambda}b_{\lambda j}+a_{i\lambda}c_{\lambda j}\right] \quad \because a,b,c \text{ are real nos}$$

$$=\sum_{\lambda=1}^{n} a_{i\lambda}b_{\lambda j}+\sum_{\lambda=1}^{n} a_{i\lambda}c_{\lambda j}$$

= element in the ith row & jth col of AB + element in ith row & jth col of AC

= element in the ith row & jth column of $(AB+AC)$

Thus $\underline{A(B+C)=AB+AC}$

(iii) $(A+B)C = AC+BC$

Let $A=[a_{ij}]_{m\times n}$ $\quad B=[b_{ij}]_{m\times n}$ $\quad C=[c_{ij}]_{n\times p}$

order of $A+B = m\times n$

order of $(A+B)C = m\times n\; n\times p$

$= \boxed{m\times p}$

order of $AC = m\times n\; n\times p = m\times p$

order of $BC = m\times n\; n\times p = m\times p$

order of $AB+BC = \boxed{m\times p}$

Now ith row of $(A+B)$ is $(a_{i1}+b_{i1}, a_{i2}+b_{i2}, a_{i3}+b_{i3} \cdots\cdots a_{in}+b_{in})$

jth column of C is $\begin{Bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{nj} \end{Bmatrix}$

The element in the ith row & jth column of $(A+B)C$ is

$$(a_{i1}+b_{i1})c_{1j}+(a_{i2}+b_{i2})c_{2j}+\cdots\cdots+(a_{in}+b_{in})c_{nj}$$

$$=\sum_{\lambda=1}^{n}(a_{i\lambda}+b_{i\lambda})c_{\lambda j}$$

$= \sum_{\lambda=1}^{n} \left[a_{i\lambda} c_{\lambda j} + b_{i\lambda} c_{\lambda j} \right]$

$= \sum_{\lambda=1}^{n} a_{i\lambda} c_{\lambda j} + \sum_{\lambda=1}^{n} b_{i\lambda} c_{\lambda j}$

= element in the ith row & jth column of AC + element in the ith row & jth column of BC

= element in the i^{th} row & j^{th} column of $AC+BC$.

Hence $(A+B)C = AC+BC$

(iv) $K(AB) = (KA)B = A(KB)$

Let. $A = [a_{ij}]_{m\times n}$ $\quad B = [b_{ij}]_{n\times p}$

order of $AB = m\times n \; n\times p = m\times p$ $\qquad$ order of $K(AB) = m\times p$

order of $(KA) = m\times n$ $\qquad$ order of $KB = n\times p$.

order of $A(KB) = m\times n \; n\times p = m\times p$

order of $K(AB) = m\times p$ = order of $K(AB)$ = order of $A(KB)$

Element in the ith row of A is $(a_{i1}, a_{i2}, \cdots a_{in})$

Element in the j^{th} column of B is $\begin{Bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{Bmatrix}$

Element in the i^{th} row & j^{th} colm of AB $= \sum_{\lambda=1}^{n} a_{i\lambda} b_{\lambda j}$

Element in the i^{th} row & j^{th} column of $K(AB) = K\left(\sum_{\lambda=1}^{n} a_{i\lambda} b_{\lambda j}\right)$

$= \sum_{\lambda=1}^{n} (K a_{i\lambda}) b_{\lambda j}$

= element in the ith row & jth column of $(KA)B$

$= \sum_{\lambda=1}^{n} a_{i\lambda} (K b_{\lambda j})$

which is element in the ith row & jth col of $A(KB)$

(i) $(A+B)^t = A^t + B^t$

$A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$

then $A' = \begin{bmatrix} a & e \\ b & f \\ c & g \\ d & h \end{bmatrix}$

Proof Let $A = [a_{ij}]_{m\times n}$ $B = [b_{ij}]_{m\times n}$

order of $A+B = m\times n$

order of $(A+B)^t = \boxed{n\times m}$

order of $A^t = n\times m$

order of $B^t = n\times m$

order of $A^t + B^t = \boxed{n\times m.}$

LH.S

Now the element in the i^{th} row & j^{th} column of $(A+B)^t$ = the element in the j^{th} row & i^{th} column of $(A+B)$

= the element in the j^{th} row & i^{th} col of A + the element in the j^{th} row & i^{th} col of B

= $a_{ji} + b_{ji}$ —— (1)

RHS Now the element in the i^{th} row & j^{th} column of $A^t + B^t$ = element in the i^{th} row & j^{th} col of A^t + element in the i^{th} row & j^{th} col of B^t

= element in the j^{th} row & i^{th} column of A + element in the j^{th} row & i^{th} column of B

= $a_{ji} + b_{ji}$ —— (2)

(1) = (2) Hence proved

(ii) $(A^t)^t = A$

Let $A = [a_{ij}]_{m\times n}$

order of $A^t = n\times m$

order of $(A^t)^t = \boxed{m\times n}$

order of $A = \boxed{m\times n}$

Now element in the i^{th} row & j^{th} column of $(A^t)^t$ = element in the j^{th} row & i^{th} column of A^t
= element in the i^{th} row & j^{th} column of A.

(iii) $(KA)^t = KA^t$

order of $KA = m \times n$
order of $KA^t = \boxed{n \times m}$
order of $(KA)^t = \boxed{n \times m}$

Let $A = [a_{ij}]_{m \times n}$

Now element in the i^{th} row & j^{th} column of $(KA)^t$ = element in the j^{th} row & i^{th} column of KA
= element in the i^{th} row & j^{th} column of KA^t

$(AB)^t = B^t A^t$

Let $A = [a_{ij}]_{m \times n}$ $B = [b_{ij}]_{n \times p}$

order of $AB = m \times n_{n \times p} = m \times p$
order of $(AB)^t = \boxed{p \times m}$

order of $B^t = p \times n$
order of $A^t = n \times m$
order of $B^t A^t = p \times n_{n \times m}$
$= \boxed{p \times m}$

Now element in the i^{th} row and j^{th} col of $(AB)^t$ = element in the j^{th} row & i^{th} column of (AB)
$= \sum_{\lambda=1}^{n} a_{j\lambda} b_{\lambda i}$ —— (I)

RHS and the element in the i^{th} row & j^{th} col of $B^t A^t$ = Sum of the products of the corresponding elements of i^{th} row of B^t & j^{th} col of A^t
= Sum of the products of the corresponding elements of i^{th} col of B & j^{th} row of A
$= \sum_{\lambda=1}^{n} a_{j\lambda} b_{\lambda i}$ —— (II)

(I) = (II) proved