Ex 3.2

Q1 Show that inverse of diagonal Matrix with all non-zero diagonal elements is a diagonal matrix.

Sol Let $A = \begin{bmatrix} d_1 & 0 & 0 \cdots 0 \\ 0 & d_2 & 0 \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots d_n \end{bmatrix}$ be a diagonal Matrix of order 'n'

and Let $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \cdots b_{1n} \\ b_{21} & b_{22} & b_{23} \cdots b_{2n} \\ \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & b_{n3} \cdots b_{nn} \end{bmatrix}$ be the inverse of diagonal Matrix 'A'.

By def of Inverse $AB = I_n$

$$\therefore \begin{bmatrix} d_1 & 0 & 0 \cdots 0 \\ 0 & d_2 & 0 \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots d_n \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \cdots b_{1n} \\ b_{21} & b_{22} \cdots b_{2n} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \cdots b_{nn} \end{bmatrix} = I_n$$

$$\begin{bmatrix} d_1b_{11} & d_1b_{12} \cdots d_1b_{1n} \\ d_2b_{21} & d_2b_{22} \cdots d_2b_{2n} \\ \vdots & \vdots \\ d_nb_{n1} & d_nb_{n2} \cdots d_nb_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \cdots 0 \\ 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots 1 \end{bmatrix}$$

$\Rightarrow d_i b_{ij} = 0 \quad \Rightarrow b_{ij} = \frac{0}{d_i} = \boxed{0}$

for $i \neq j = 1, 2, \ldots n$

&

$\Rightarrow d_i b_{ii} = 1 \quad \Rightarrow b_{ii} = \boxed{\frac{1}{d_i}}$

for $i = 1, 2, \ldots n$

Hence.

$B = \begin{bmatrix} \frac{1}{d_1} & 0 & 0 \cdots 0 \\ 0 & \frac{1}{d_2} & 0 \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots \frac{1}{d_n} \end{bmatrix}$ is the diagonal Matrix

$\therefore$ Inverse of diagonal Matrix A is diagonal Matrix B.

Q2 Show that inverse of a scalar Matrix is a scalar Matrix.

Sol Let $A = \begin{bmatrix} c & 0 & 0 \cdots 0 \\ 0 & c & 0 \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots c \end{bmatrix}$ be a Scalar Matrix of order 'n'

and

$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \cdots b_{1n} \\ b_{21} & b_{22} & b_{23} \cdots b_{2n} \\ \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & b_{n3} \cdots b_{nn} \end{bmatrix}$ be the inverse of Scalar Matrix A

By def of Inverse

$$AB = I_n$$

$$\therefore \begin{bmatrix} c & 0 & 0 \cdots 0 \\ 0 & c & 0 \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots c \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \cdots b_{1n} \\ b_{21} & b_{22} \cdots b_{2n} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \cdots b_{nn} \end{bmatrix} = I_n$$

$$\begin{bmatrix} b_{11}c & b_{12}c \cdots b_{1n}c \\ b_{21}c & b_{22}c \cdots b_{2n}c \\ \vdots & \vdots \\ b_{n1}c & b_{n2}c \cdots b_{nn}c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \cdots 0 \\ 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots 1 \end{bmatrix}$$

$\Rightarrow b_{ij}c = 0 \quad \Rightarrow b_{ij} = \frac{0}{c} = \boxed{0}$

for $i \neq j = 1, 2, \ldots n$

& $b_{ii}c = 1 \quad \Rightarrow b_{ii} = \boxed{\frac{1}{c}}$

for $i = 1, 2, \ldots n$

Hence

$B = \begin{bmatrix} \frac{1}{c} & 0 & 0 \cdots 0 \\ 0 & \frac{1}{c} & 0 \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots \frac{1}{c} \end{bmatrix}$

is the Scalar Matrix

$\therefore$ Inverse of Scalar Matrix is Scalar.

(2) (i) $(A^n)^{-1} = (A^{-1})^n$ To Prove

Sol: $(A^n)^{-1} = (A \cdot A \cdot A \cdots n \text{ times})^{-1}$

$= A^{-1}A^{-1}A^{-1} \cdots n \text{ times}$

$(A^n)^{-1} = (A^{-1})^n$ Proved.

(iv) $(\bar{A})^{-1} = \overline{(A^{-1})}$

$(\bar{A})(\overline{A^{-1}}) = \overline{(AA^{-1})} = \bar{I} = I$

Pre Multiply by $(\bar{A})^{-1}$

$(\bar{A})^{-1}(\bar{A})(\overline{A^{-1}}) = (\bar{A})^{-1} I$

$I(\overline{A^{-1}}) = (\bar{A})^{-1}$

$(\overline{A^{-1}}) = (\bar{A})^{-1}$

(iii) $(A^{-1})^t = (A^t)^{-1}$ To Prove

Sol $(A^{-1})^t(A^t) = (AA^{-1})^t = I^t = I$

$[(A^{-1})^t(A^t)](A^t)^{-1} = I(A^t)^{-1}$ Post x by $(A^t)^{-1}$

$(A^{-1})^t(I) = (A^t)^{-1}$

$(A^{-1})^t = (A^t)^{-1}$ proved

(ii) $(KA)^{-1} = K^{-1}A^{-1}$ To Prove

Sol $(KA)(K^{-1}A^{-1})$

$= (AK)(K^{-1}A^{-1})$

$= A(KK^{-1})A^{-1}$

$= A\,I\,A^{-1}$

$= AA^{-1}$

$= I$

$(KA)^{-1}[(KA)(K^{-1}A^{-1})] = (KA)^{-1} I$ Pre Multiply by $(KA)^{-1}$

$(I)\,K^{-1}A^{-1} = (KA)^{-1}$

$K^{-1}A^{-1} = (KA)^{-1}$

(v) $(\overline{A^t})^{-1} = \overline{(A^{-1})^t}$

$(\overline{A^t})\overline{(A^{-1})^t} = \overline{A^t(A^{-1})^t} = \overline{(A^{-1}A)^t} = \overline{(I)^t} = \bar{I} = I$

Pre multiply both sides by $(\overline{A^t})^{-1}$

$(\overline{A^t})^{-1}(\overline{A^t})\overline{(A^{-1})^t} = (\overline{A^t})^{-1} I$

$\overline{(A^{-1})^t} = (\overline{A^t})^{-1}$ proved.

Q4 Given that $A^2B = B^2A$ —(i) and $A^3 = B^3$ —(ii), A, B are diff

Suppose A^2+B^2 is invertible

then $A-B = I(A-B)$ $\because I$ is identity Matrix

$= (A^2+B^2)^{-1}(A^2+B^2)(A-B)$ $\because I = (A^2+B^2)^{-1}(A^2+B^2)$

$= (A^2+B^2)^{-1}(A^3 - A^2B + B^2A - B^3)$

$= (A^2+B^2)^{-1}(0)$ using (i) & (ii)

$A-B = 0$

$\Rightarrow A = B$, a contradiction as A & B are given diff matrices.

Hence our supposition is wrong and A^2+B^2 is not invertible.

Note

$(A^t)^{-1} = (A^{-1})^t$

$\overline{(A^{-1})} = (\bar{A})^{-1}$

$(A^{-1})^n = (A^n)^{-1}$

$\overline{(A^t)} = (\bar{A})^t$

$(A^t)^n = (A^n)^t$

$\overline{(A^n)} = (\bar{A})^n$

All these inverse, transpose conjugate, exponent are interchangeable.

Q3. If A is invertible & $AB = 0$ show that $B = 0$

Sol ∵ A is invertible so A^{-1} exists

$$AB = 0 \quad \text{given}$$
$$A^{-1}AB = A^{-1}0$$
$$IB = 0$$
$$B = 0$$

Elementary Row Operations (ERO)

i) These operations on a matrix are called ERO

(i) Interchange of any two rows denoted by 'R_{ij}' i.e i^{th} & j^{th} rows are interchanged.

(ii) Multiplication of a row by a scalar denoted by $K(R_i)$ i.e each element of i^{th} row is multiplied by K.

(iii) Addition of a multiple of one row to any other row, denoted by $(K)R_i + R_j$

Row Equivalent Matrices:

A matrix B is said to be row equivalent to a matrix A of same order if B can be obtained from A by applying finite no of ERO on A.

$$B \overset{R}{\sim} A \qquad B \text{ is row equivalent to } A.$$

Elementary Matrix: The matrix E obtained by applying one ERO to I is called an Elementary Matrix.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K \end{bmatrix} \quad \text{by } KR_3$$

Echelon Form of a Matrix

A matrix is said to be in Echelon form if it has the following structure

i) All the zero rows are below the non-zero rows of A.

ii) The number of zeros occuring before the first nonzero entry in each non-zero row is greater than the number of zeros that appear before the first nonzero element in any preceeding row.

For example

$$\begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 0, & -1 & -2 \end{bmatrix} \qquad A = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 & 9 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A, B, C are not in Row Echelon Form.
D, E, F, G, H are in Reduced Echelon Form.
Rest are Echelon Form

$$D = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The first nonzero element in each row of an Echelon Matrix Form is called PIVOT element of that row. A column containg PIVOT is called PIVOT Column.

Reduced Echelon Form has the following structure

$$X = \begin{bmatrix} 0 & 5 & 2 & 1 \\ 0 & 0 & 9 & -1 \\ 0 & 0 & 0 & 6 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

X is in Echelon
Y is in Reduced Echelon

i) It is in Echelon Form

ii) Pivot element of each row is '1' i.e first nonzero element is '1' in each row.

iii) Every entry in the pivot column is zero except the pivot element '1'

Note It becomes easy to reduce a matrix in Echelon form if we obtain Pivot element to be '1', Also some authors include this condition in Echelon form instead of Reduced Echelon form.
Echelon form is different for a matrix depending upon the sequence of ERO applied but Reduced Echelon form is the same.

Th If a square matrix A is reduced to the identity matrix by a sequence of elementary row operations, the same sequence of operations performed on the identity matrix produces the inverse of A.

Proof

$$I \overset{R}{\sim} A$$

$$\Rightarrow I = (E_r E_{r-1} E_{r-2} \cdots E_2 E_1) A$$

where E_i are suitable elementary matrices.

Post Multiply both sides by A^{-1}

$$I A^{-1} = (E_r E_{r-1} E_{r-2} \cdots E_2 E_1) A A^{-1}$$

$$A^{-1} = (E_r E_{r-1} E_{r-2} \cdots E_2 E_1) I$$

x ———— x

Q4 Find the inverse of

(i) $\begin{bmatrix} 1 & 0 & K \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ over R

Matrix A. $\quad\quad$ I_3

$$\begin{bmatrix} 1 & 0 & K \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reducing 'A' to Identity matrix by ERO

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -K \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{by } -KR_3+R_1$$

I_3

Hence $\begin{bmatrix} 1 & 0 & -K \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the inverse of 'A'

(ii) $A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ $\qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ $\qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ by R_{13}

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\qquad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ by R_{23}

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\qquad \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ by $(-1)R_2$

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\qquad \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ by $(-1)R_3$

I_3

Hence $\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ is inverse of matrix A.

x —————— x

(iii) $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$ $\qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$ $\qquad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by $(-1)R_1$

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -6 \\ 0 & 6 & -7 \end{bmatrix}$ $\qquad \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$ by $-2R_1+R_2$, $-4R_1+R_3$

$$\sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -6 \\ 0 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \quad -R_2+R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 5 & -6 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \quad R_{23}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 3 & -2 & 2 \\ 2 & -1 & 1 \\ -8 & 6 & -5 \end{bmatrix} \quad \begin{matrix} 2R_2+R_1 \\ -5R_2+R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 3 & -2 & 2 \\ 2 & -1 & 1 \\ 8 & -6 & 5 \end{bmatrix} \quad (-1)R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix} \quad \begin{matrix} -R_3+R_1 \\ R_3+R_2 \end{matrix}$$

$$I_3 \qquad \text{Hence} \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix} = A^{-1}$$

(iv)

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad -2R_1+R_3$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad R_{13}$$

$$\sim_R \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & +1 \end{bmatrix} \quad \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & 0 & -2 \end{bmatrix} \quad -2R_1+R_3$$

$$\sim_R \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} -2 & 0 & 1 \\ 5 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \quad R_{23}$$

$$\sim_R \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} -2 & 0 & 1 \\ 5 & 0 & -2 \\ -10 & 1 & 4 \end{bmatrix} \quad -2R_2+R_3$$

$$\sim_R \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} -2 & 0 & 1 \\ 5 & 0 & -2 \\ 10 & -1 & -4 \end{bmatrix} \quad (-1)R_3$$

$$\sim_R \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_3} \quad \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix} \quad \begin{matrix} R_3+R_1 \\ -R_3+R_2 \end{matrix}$$

Hence $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix} = A^{-1}$

(vi) $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\sim_R \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{I_3} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_{13}$$

Hence $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A^{-1}$

v)
$$A=\begin{pmatrix} i & -1 & 2i \\ 2 & 0 & 2 \\ -1 & 0 & 1 \end{pmatrix} \qquad I_3=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & i & 2 \\ 2 & 0 & 2 \\ -1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (-i)R_1$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & i & 2 \\ 0 & -2i & -2 \\ 0 & i & 3 \end{pmatrix} \qquad \begin{pmatrix} -i & 0 & 0 \\ 2i & 1 & 0 \\ -i & 0 & 1 \end{pmatrix} \qquad \begin{matrix} -2R_1+R_2 \\ R_1+R_3 \end{matrix}$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & i & 2 \\ 0 & 1 & -i \\ 0 & i & 3 \end{pmatrix} \qquad \begin{pmatrix} -i & 0 & 0 \\ -1 & i/2 & 0 \\ -i & 0 & 1 \end{pmatrix} \qquad \left(\frac{i}{2}\right)R_2$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -i \\ 0 & 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -1 & i/2 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix} \qquad \begin{matrix} -iR_2+R_1 \\ -iR_2+R_3 \end{matrix}$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -i \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -1 & i/2 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \qquad \frac{1}{2}R_3$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & \frac{1}{4} & -\frac{1}{2} \\ -1 & \frac{3i}{4} & \frac{i}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \qquad \begin{matrix} -R_3+R_1 \\ iR_3+R_2 \end{matrix}$$

I_3

Hence
$$A^{-1}=\begin{pmatrix} 0 & \frac{1}{4} & -\frac{1}{2} \\ -1 & \frac{3i}{4} & \frac{i}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

(5) (i) $\begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & 3 & -4 & 4 \\ 0 & 7 & -7 & 6 \end{pmatrix}$ $\quad -2R_1+R_2$, $-3R_1+R_3$

$\sim \begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -\frac{4}{3} & \frac{4}{3} \\ 0 & 7 & -7 & 6 \end{pmatrix}$ $\quad \frac{1}{3}R_2$

$\sim \begin{pmatrix} 1 & 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & -4/3 & \frac{4}{3} \\ 0 & 0 & 7/3 & -\frac{10}{3} \end{pmatrix}$ $\quad 2R_2+R_1$, $-7R_2+R_3$

$\sim \begin{pmatrix} 1 & 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & -4/3 & 4/3 \\ 0 & 0 & 1 & -10/7 \end{pmatrix}$ $\quad \frac{3}{7}R_3$

$\sim \begin{pmatrix} 1 & 0 & 0 & 15/7 \\ 0 & 1 & 0 & -4/7 \\ 0 & 0 & 1 & -10/7 \end{pmatrix}$ $\quad \frac{4}{3}R_3+R_2$, $-\frac{1}{3}R_3+R_1$ Required Reduced Echelon Form

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$A = \begin{pmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{pmatrix}$

$\sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 2 & 3 & 2 & -1 \end{pmatrix}$ $\quad R_{12}$

$$\underset{\sim}{R}\begin{pmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} \\ 0 & 1 & 3 & -2 \\ 2 & 3 & 2 & -1 \end{pmatrix} \quad \frac{1}{2}R_1$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & -4 \end{pmatrix} \quad -2R_1+R_3$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & 0 & -7/2 & 5/2 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} -\frac{1}{2}R_2+R_1 \\ -2R_2+R_3 \end{matrix}$$

Required Reduced Echelon Form

(iii) $$A=\begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 3 \end{pmatrix}$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 5 & -12 & 0 \end{pmatrix} \quad \begin{matrix} -2R_1+R_2 \\ -3R_3+R_3 \end{matrix}$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & \frac{1}{3} \\ 0 & 0 & 5 & -12 & 0 \end{pmatrix} \quad \frac{1}{3}R_2$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1/3 \\ 0 & 0 & 0 & -2 & -5/3 \end{pmatrix} \quad -5R_2+R_3$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1/3 \\ 0 & 0 & 0 & 1 & 5/6 \end{pmatrix} \quad \frac{-1}{2}R_3$$

Echelon Form.

(iv)
$$A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & -11 & 5 & -3 \\ 0 & -11 & 5 & -3 \end{pmatrix} \quad \begin{matrix} -2R_3+R_3 \\ -4R_1+R_4 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -\frac{5}{11} & \frac{3}{11} \\ 0 & -11 & 5 & -3 \\ 0 & -11 & 5 & -3 \end{pmatrix} \quad \frac{1}{11}R_2$$

$$\begin{pmatrix} 1 & 0 & \frac{4}{11} & 13/11 \\ 0 & 1 & -5/11 & 3/11 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} -3R_2+R_1 \\ 11R_2+R_3 \\ 11R_2+R_4 \end{matrix}$$

Required Reduced Echelon Form

Q6
$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 1 & 2 \end{pmatrix} \quad -3R_1+R_2$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & -5 & -1 \end{pmatrix} \quad R_{23}$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 0 & -3\\ 0 & 1 & 2\\ 0 & 0 & 9\end{bmatrix} \quad \begin{matrix}5R_2+R_3\\ -2R_2+R_1\end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 0 & -3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix} \quad \frac{1}{9}R_3$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \quad \begin{matrix}3R_3+R_1\\ -2R_3+R_2\end{matrix}$$

Hence $A \overset{R}{\sim} I_3$

Elementary Column Operations.

Equivalent Matrices:

A $m \times n$ matrix B is said to be equivalent to an $m \times n$ matrix A if B can be obtained from A by applying some elementary row & column operations on A. We denote it as $B \sim A$.

$$\text{If } B \sim A \text{ then } B = PAQ.$$

- where P and Q are non-singular matrices of order m & n.

'P' is obtained from I_m by some row operations applied on A to get B.

'Q' is obtained from I_n by the same column operations applied on A to get B.

Normal or Canonical form of a matrix.

A matrix is said to be in normal or canonical form when it has the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

where I_r is Identity matrix of order 'r' & the remaining submatrices are zero matrices. The following are normal matrices.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = \begin{bmatrix} I_3 & 0 \end{bmatrix} \qquad \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\& \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank of a Matrix

No of non-zero rows in Echelon or Reduced Echelon Form of a matrix is called Rank of a Matrix.

No of non-zero columns in Echelon or reduced Echelon Form of a matrix is called Column Rank of a matrix.

The row rank & the column rank of a matrix are equal.

Q7
$$A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 0 & -16 \\ 0 & 9 \end{bmatrix} \quad \begin{matrix} -5R_1+R_3 \\ 2R_1+R_4 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & -16 \\ 0 & 9 \end{bmatrix} \quad -\frac{1}{2} R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} 16R_2+R_3 \\ -9R_2+R_4 \end{matrix}$$

No of non zero Rows = 2

So Rank = 2

3.2-16

(ii)
$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{bmatrix}$$

$$\sim_R \begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 6 \\ 0 & 3 & -3 \\ 0 & 6 & -5 \end{bmatrix} \quad \begin{matrix} -2R_1 + R_2 \\ +2R_1 + R_3 \\ +R_1 + R_4 \end{matrix}$$

$$\sim_R \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 3 & -3 \\ 0 & 6 & -5 \end{bmatrix} \quad -\frac{1}{3} R_2$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 7 \end{bmatrix} \quad \begin{matrix} -3R_2 + R_3 \\ -6R_2 + R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 7 \end{bmatrix} \quad \frac{1}{3} R_3$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad -7R_3 + R_4$$

No of non zero rows are 3 (Ist, 2nd, IIIrd)

So Rank $A = 3$

(iii) $$\begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & -1 & 1 \end{pmatrix} \quad \begin{array}{l} \\ -R_1 + R_2 \\ -2R_1 + R_3 \\ -3R_1 + R_4 \end{array}$$

$$\sim \begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \\ \\ 3R_2 + R_3 \\ R_2 + R_4 \end{array}$$

Non zero rows are Ist & 2nd

Hence Rank = 2

(iv) $$\begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & 2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 1 & 4 & -1 & -1 \\ 0 & 1 & 1 & -4 & 5 \end{pmatrix} \quad \begin{array}{l} \\ -R_1 + R_2 \\ -R_1 + R_3 \\ -2R_1 + R_4 \end{array}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & -2 & -2 & 4 \end{bmatrix} \quad \begin{matrix} -R_2+R_3 \\ -R_2+R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad 2R_3+R_4$$

3 Nonzero Rows 1st, 2nd, 3rd

Hence Rank = 3.

(8) Matrix A   Row Operation   Col operation.

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & -4 & 1 \\ 0 & 3 & 2 \end{bmatrix}_{3\times 3} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 3 \\ 0 & -2 & -5 \\ 0 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} -2R_1+R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -3 \\ 0 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} +1 & 0 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_3+R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2+R_1$$

3.2—19

A   Row Operations   Col operations.

9.(iii)

$$\begin{bmatrix}1 & -1 & 3\\ 2 & -4 & 1\\ 0 & 3 & 2\end{bmatrix}_{3\times3} \quad \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}_{3\times3} \quad \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}_{3\times3}$$

$$\begin{bmatrix}1 & -1 & 3\\ 0 & -2 & -5\\ 0 & 3 & 2\end{bmatrix} \quad \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \quad \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \text{ by } -2R_1+R_2$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & -2 & -5\\ 0 & 3 & 2\end{bmatrix} \quad \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \quad \begin{bmatrix}1 & 1 & -3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \text{ by } 1C_1+C_2,\ -3C_1+C_3$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -3\\ 0 & 3 & 2\end{bmatrix} \quad \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 1\\ 0 & 0 & 1\end{bmatrix} \quad \begin{bmatrix}1 & 1 & -3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \text{ by } R_3+R_2$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -3\\ 0 & 0 & 11\end{bmatrix} \quad \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 1\\ 6 & -3 & -2\end{bmatrix} \quad \begin{bmatrix}1 & 1 & -3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \ -3R_2+R_3$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 11\end{bmatrix} \quad \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 1\\ 6 & -3 & -2\end{bmatrix} \quad \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 3\\ 0 & 0 & 1\end{bmatrix} \ 3C_2+C_3$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \quad \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 1\\ 6 & -3 & -2\end{bmatrix} \quad \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 3/11\\ 0 & 0 & 1/11\end{bmatrix} \ \frac{1}{11}C_3$$

$$I_3 \qquad\qquad P \qquad\qquad Q$$

Hence Normal form of $\begin{bmatrix}1 & -1 & 3\\ 2 & -4 & 1\\ 0 & 3 & 2\end{bmatrix}$ is I_3

x———————x

3.2-20

Normal or Canonical form :-

A matrix is said to be in normal form when it has the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where I_r is identity matrix of order 'r' and remaining submatrices are zero matrices.

e.g $\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$

$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = \begin{bmatrix} I_3 & O_{3\times1} \end{bmatrix}$

Q9 Reduce in Canonical form

i) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}_{2\times2}$

	For Row Operation	For col operation	
$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	
$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	by R_{12}
$\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	by $-2R_1+R_2$
$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$	by $-2C_1+C_2$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & -\frac{1}{3} \end{bmatrix}$	by $\frac{1}{-3}C_2$
I_2	P	Q	

Note P, Q are not unique. i.e may be diff, if we operate in a different manner.

(ii) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times3}$

$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	by R_{12}
$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	by $-2R_1+R_2$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	by $-C_1+C_2$, by $-C_1+C_3$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$-R_2$

by $-C_2+C_3$

$\left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} I_2 & O_{2\times1} \end{bmatrix}$ $\quad P \quad$ Q

3.2-21

Q(iv)

$$\begin{array}{ccc} A & I_3 & I_4 \\ \begin{bmatrix}1&2&0&-1\\3&4&1&2\\-2&3&2&5\end{bmatrix}_{3\times4} & \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}_{3\times3} & \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}_{4\times4}\end{array}$$

$$\begin{bmatrix}1&2&0&-1\\0&-2&1&5\\0&7&2&3\end{bmatrix} \begin{bmatrix}1&0&0\\-3&1&0\\2&0&1\end{bmatrix} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \quad \text{by } -3R_1+R_2,\ 2R_1+R_3$$

$$\begin{bmatrix}1&0&0&0\\0&-2&1&5\\0&7&2&3\end{bmatrix} \begin{bmatrix}1&0&0\\-3&1&0\\2&0&1\end{bmatrix} \begin{bmatrix}1&-2&0&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \quad \text{by } -2C_1+C_2,\ 1C_1+C_4$$

$$\begin{bmatrix}1&0&0&0\\0&1&-2&5\\0&2&7&3\end{bmatrix} \begin{bmatrix}1&0&0\\-3&1&0\\2&0&1\end{bmatrix} \begin{bmatrix}1&0&-2&1\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix} \quad \text{by } C_{23}$$

$$\begin{bmatrix}1&0&0&0\\0&1&-2&5\\0&0&11&-7\end{bmatrix} \begin{bmatrix}1&0&0\\-3&1&0\\8&-2&1\end{bmatrix} \begin{bmatrix}1&0&-2&1\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix} \quad \text{by } -2R_2+R_3$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&11&-7\end{bmatrix} \begin{bmatrix}1&0&0\\-3&1&0\\8&-2&1\end{bmatrix} \begin{bmatrix}1&0&-2&1\\0&0&1&0\\0&1&2&-5\\0&0&0&1\end{bmatrix} \quad 2C_2+C_3,\ -5C_2+C_4$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-7\end{bmatrix} \begin{bmatrix}1&0&0\\-3&1&0\\8&-2&1\end{bmatrix} \begin{bmatrix}1&0&-2/11&1\\0&0&1/11&0\\0&1&2/11&-5\\0&0&0&1\end{bmatrix} \quad \frac{1}{11}C_3$$

$$\begin{array}{ccc} \left[\begin{array}{ccc:c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right] & \begin{bmatrix}1&0&0\\-3&1&0\\8&-2&1\end{bmatrix} & \begin{bmatrix}1&0&&-14/11\\0&0&&7/11\\0&1&&-41/11\\0&0&&1\end{bmatrix} \\ [I_3 \quad 0] & P & Q \end{array} \quad \text{by } 7C_3+C_4$$

$$\frac{14}{11}-5=\frac{-41}{11}$$

Hence the required normal form is $[I_3 \quad 0]$

x ———————— x

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 3 & 1 \\ 2 & -2 & 1 & 0 & 2 \\ 1 & 1 & -1 & -3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

4×5

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 2 & -2 & -4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} -R_1+R_2 \\ -2R_1+R_3 \\ -R_1+R_4 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 2 & 2 & -4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} C_1+C_2 \\ -C_1+C_3 \\ -C_1+C_4 \\ -C_1+C_5 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & 2 & 2 & -4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad C_{23}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -8 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 6 & 0 \\ -1 & 1 & 0 & 0 \\ -3 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} R_2+R_3 \\ -2R_2+R_4 \end{array}$$

3.2 - 23

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -8 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -3 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad -2C_2+C_4$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -8 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 \\ -3 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad R_{34}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{1}{2} & -1 & 0 & \frac{1}{2} \\ -3 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \frac{1}{2}R_3$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{1}{2} & -1 & 0 & \frac{1}{2} \\ -3 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & 5 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{matrix} 4C_3+C_4 \\ C_3+C_5 \end{matrix}$$

$$\begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix} \qquad P \qquad Q$$