Orthogonal Matrices

A square matrix A over the field R is called Orthogonal if $A^tA = I = AA^t$

OR $A^t = A^{-1}$

A^t is inverse of A $\because A^tA = I$

From the def it follows that

i) A is orthogonal iff A^{-1} is orthogonal

ii) If A is orthogonal then A is symmetric iff $A^2 = I$

iii) If A & B are orthogonal then AB and BA are also orthogonal.

iv) If A is orthogonal then the col vectors are of unit length and their dot products are zero i.e Columns of A form an orthonormal set

v) If A is orthogonal then the rows of A form an orthonormal set.

vi) A^{-1} is orthogonal iff A^t is orthogonal.

Theorem The following conditions for a square Matrix A are equivalent:

i) A is orthogonal

ii) The rows of A form an orthonormal set.

iii) The col of A form an orthonormal set.

Proof Let $A = [a_{ij}]_{n\times n}$ be an Orthogonal Matrix i.e $AA^t = I = A^tA$

Let R_i & C_i denote the i^{th} row & i^{th} col of A where $1 \le i \le n$

Let c_{ij} = (i,j) element of AA^t

= Sum of products of the corresponding elements in the i^{th} row of A and j^{th} col of A^t.

= Sum of products of the corresponding elements in the i^{th} row of A and j^{th} row of A

$= a_{i1}a_{j1} + a_{i2}a_{j2} + \dots a_{in}a_{jn}$

i^{th} row of $A = (a_{i1}, a_{i2}, \dots a_{in})$
j^{th} row of $A = (a_{j1}, a_{j2}, \dots a_{jn})$
for detail see examples

$c_{ij} = \langle R_i, R_j \rangle$ ——— (1)
(I.P of two rows)

Now $\overset{\text{Matrix}}{[c_{ij}]} = AA^t$

$[c_{ij}] = I$ $\qquad$ $\because A$ is orthogonal so $AA^t = I$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

when $i=j$ $c_{ij} = 1$
when $i \ne j$ $c_{ij} = 0$

$\therefore \langle R_i, R_j \rangle = c_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$

$\because c_{ij} = \langle R_i, R_j \rangle$

$\therefore \langle R_i, R_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$ $\Rightarrow$ Rows of A are Orthonormal

$\Rightarrow$ (i) $\Rightarrow$ (ii) (from statement of Theorem)

Conversely if rows of A form an orthonormal set, then

$$\langle R_i, R_j\rangle = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$$

$\because \langle R_i, R_j\rangle = c_{ij}$ $\therefore c_{ij} = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$

$\overset{\text{Matrix}}{\Rightarrow} [c_{ij}] = I$

$\because AA^t = [c_{ij}]$ $\therefore AA^t = I \Rightarrow A$ is orthogonal Matrix

So (ii) $\Rightarrow$ (i) Hence (i) $\Leftrightarrow$ (ii)

Now Consider the Matrix $A^tA = [d_{ij}]$

$d_{ij} = (i,j)$ element of A^tA

= Sum of the products of the corresponding elements in i^{th} row of A^t and j^{th} col of A.

= Sum of the products of the corresponding elements in i^{th} col of A and j^{th} col of A

$= a_{1i}a_{1j} + a_{2i}a_{2j} + \cdots a_{ni}a_{nj}$

$\because i^{th}$ row of $A^t = [a_{i1}\ a_{i2} \cdots a_{in}]$

i^{th} col of $A = \begin{bmatrix}a_{1i}\\ a_{2i}\\ \vdots\\ a_{ni}\end{bmatrix}$

j^{th} col of $A = \begin{bmatrix}a_{1j}\\ a_{2j}\\ a_{nj}\end{bmatrix}$

= Inner product of i^{th} & j^{th} col matrices

$d_{ij} = \langle C_i, C_j\rangle$

$[d_{ij}] = A^tA = I$ $\because A$ is orthogonal Matrix

$\Rightarrow d_{ij} = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$

$\because d_{ij} = \langle C_i, C_j\rangle$

$\therefore \langle C_i, C_j\rangle = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$ $\Rightarrow$ cols of A C_i & C_j are orthonormal

$\Rightarrow$ (i) $\Rightarrow$ (iii)

Conversely if columns of A form orthonormal set, then

$$\langle C_i, C_j\rangle = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$$

$\because \langle C_i, C_j\rangle = d_{ij}$ $\therefore d_{ij} = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$

$\Rightarrow [d_{ij}] = I$

$\because [d_{ij}] = A^tA$ $\therefore A^tA = I$ $\Rightarrow A$ is orthogonal Matrix

So (iii) $\Rightarrow$ (i) Hence (i) $\Leftrightarrow$ (iii)

Now (i) $\Leftrightarrow$ (ii) & (i) $\Leftrightarrow$ (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii)

Note: As (ii) $\Leftrightarrow$ (iii) so A is orthogonal iff A^t is orthogonal

Example 13 Show that the rows (columns) of the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ form an orthonormal set.

Sol To show that the rows (colmns) of the matrix A form an orthonormal set we have only to show that A is orthogonal (Th 7.12 on page 277)
(If A is orthogonal then $AA^t = I$)

$$\therefore AA^t = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta+\sin^2\theta+0 & \cancel{\cos\theta\sin\theta} - \cancel{\sin\theta\cos\theta}+0 & 0+0+0 \\ \cancel{\sin\theta\cos\theta} - \cancel{\cos\theta\sin\theta}+0 & \sin^2\theta+\cos^2\theta+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Thus A is orthogonal
So Rows of A form an orthonormal set.

Example 14 Find an orthogonal matrix A whose first row is $\left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]$

Sol First Method Let $R_1 = \boxed{\left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]}$

A vector (row) orthogonal to R_1 is R_2^*

$$R_2^* = \begin{vmatrix} e_1 & e_2 \\ \frac{1}{3} & \frac{2}{3} \end{vmatrix} + \begin{vmatrix} e_2 & e_3 \\ \frac{2}{3} & \frac{2}{3} \end{vmatrix} + \begin{vmatrix} e_3 & e_1 \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix}$$

$$= \cancel{e_1\left(\tfrac{2}{3}\right)} - e_2\left(\tfrac{1}{3}\right) + e_2\left(\tfrac{2}{3}\right) - e_3\left(\tfrac{2}{3}\right) + e_3\left(\tfrac{1}{3}\right) - \cancel{e_1\left(\tfrac{2}{3}\right)}$$

$$= e_2\left(\tfrac{2}{3}-\tfrac{1}{3}\right) + e_3\left(-\tfrac{2}{3}+\tfrac{1}{3}\right) = e_2\left(\tfrac{1}{3}\right) + e_3\left(-\tfrac{1}{3}\right)$$

$$R_2^* = \left(0, \tfrac{1}{3}, -\tfrac{1}{3}\right)$$

$$R_2 = \frac{R_2^*}{\|R_2^*\|} = \frac{\left(0,\frac{1}{3},-\frac{1}{3}\right)}{\sqrt{2}/3} = \frac{3}{\sqrt{2}}\left(0,\tfrac{1}{3},-\tfrac{1}{3}\right) = \boxed{\left(0,\tfrac{1}{\sqrt{2}},-\tfrac{1}{\sqrt{2}}\right)}$$

A vector (row) orthogonal to R_1 & R_2^*

$$R_3^* = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = e_1\left(-\tfrac{2}{9}-\tfrac{2}{9}\right) - e_2\left(-\tfrac{1}{9}\right) + e_3\left(\tfrac{1}{9}\right)$$

$$= -\tfrac{4}{9}e_1 + \tfrac{1}{9}e_2 + \tfrac{1}{9}e_3$$

$$= \left(-\tfrac{4}{9}, \tfrac{1}{9}, \tfrac{1}{9}\right)$$

$$R_3 = \frac{R_3^*}{\|R_3^*\|} = \frac{\left(-\frac{4}{9},\frac{1}{9},\frac{1}{9}\right)}{\frac{\sqrt{18}}{9}} = \frac{9}{3\sqrt{2}}\left(-\tfrac{4}{9},\tfrac{1}{9},\tfrac{1}{9}\right) = \boxed{\left(\tfrac{-4}{3\sqrt{2}},\tfrac{1}{3\sqrt{2}},\tfrac{1}{3\sqrt{2}}\right)}$$

$$\therefore \text{Orthogonal Matrix } A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{-4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{bmatrix}$$

(To Check Find A^tA. we get $A^tA = I$)

Example 14 — 2nd Method

Let $R_1 = \boxed{\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)}$

A vector orthogonal to R_1 is R_2^*

$$R_2^* = \begin{vmatrix} e_1 & e_2 \\ \frac{1}{3} & \frac{2}{3} \end{vmatrix} + \begin{vmatrix} e_2 & e_3 \\ \frac{2}{3} & \frac{2}{3} \end{vmatrix} + \begin{vmatrix} e_3 & e_1 \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix}$$

$$= e_1\left(\frac{2}{3}\right) - e_2\left(\frac{1}{3}\right) + e_2\left(\frac{2}{3}\right) - e_3\left(\frac{2}{3}\right) + e_3\left(\frac{1}{3}\right) - e_1\left(\frac{2}{3}\right)$$

$$R_2^* = e_2\left(\frac{1}{3}\right) + e_3\left(-\frac{1}{3}\right) = \left(0, \frac{1}{3}, -\frac{1}{3}\right)$$

$$\|R_2^*\| = \sqrt{0 + \frac{1}{9} + \frac{1}{9}} = \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3}$$

$$R_2 = \frac{R_2^*}{\|R_2^*\|} = \frac{\left(0, \frac{1}{3}, -\frac{1}{3}\right)}{\frac{\sqrt{2}}{3}} = \frac{3}{\sqrt{2}}\left(0, \frac{1}{3}, -\frac{1}{3}\right) = \boxed{\left(0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)}$$

A vector orthogonal to R_1 & R_2 is R_3^*

$$R_3^* = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{vmatrix} = e_1\left(\frac{-2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}}\right) - e_2\left(\frac{-1}{3\sqrt{2}}\right) + e_3\left(\frac{1}{3\sqrt{2}}\right)$$

$$= \left(\frac{-4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right)$$

$$R_3 = \frac{R_3^*}{\|R_3^*\|} = \frac{\left(\frac{-4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right)}{1} = \boxed{\frac{-4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}}$$

Hence orthogonal Matrix $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{-4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{bmatrix}$

Note
We can check our Answer by $\langle R_1, R_2\rangle = 0 = \langle R_1, R_3\rangle$
$0 = \langle R_2, R_3\rangle$
$\langle R_1, R_1\rangle = \langle R_2, R_2\rangle = 1 = \langle R_3, R_3\rangle$
OR $AA^t = I$

3rd Method

Let $V_1 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ & $\omega_2 = (x, y, z)$

Let ω_2 is orthogonal to V_1 $\therefore \langle V_1, \omega_2\rangle = 0$

$\Rightarrow \frac{x}{3} + \frac{2y}{3} + \frac{2z}{3} = 0$ or $x + 2y + 2z = 0$ —(1)

If we take $\boxed{x = 0}$ then $2y + 2z = 0 \Rightarrow \boxed{z = -y}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ -y \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \therefore \omega_2 = (x, y, z) = (0, 1, -1)$$

$$V_2 = \frac{\omega_2}{\|\omega_2\|} = \frac{(0, 1, -1)}{\sqrt{1+1}} = \boxed{\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)}$$

$V_1 \Rightarrow$ order of A is 3×3
$\because A$ must be square.
V_1 is first row of A
V_2 is 2nd row of A
(we may take $y = 0$ or $z = 0$)

Now let $\omega_3 = (x, y, z)$ be orthogonal to V_1 & V_2

$\therefore \langle V_1, \omega_3\rangle = 0 \Rightarrow \frac{x}{3} + \frac{2y}{3} + \frac{2z}{3} = 0 \Rightarrow x + 2y + 2z = 0$ —(II)

$\langle V_2, \omega_3\rangle = 0 \Rightarrow \frac{y}{\sqrt{2}} - \frac{z}{\sqrt{2}} = 0 \Rightarrow y - z = 0 \Rightarrow y = z$ Put in (II)
$\Rightarrow x = -4z$ from (II)

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4z \\ z \\ z \end{bmatrix} = z\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \quad \therefore \omega_3 = (-4, 1, 1)$$

$$V_3 = \frac{\omega_3}{\|\omega_3\|} = \frac{(-4, 1, 1)}{\sqrt{18}} = \left(\frac{-4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right)$$

Hence $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{-4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{bmatrix}$