Eigen Values and Eigen Vectors

Let A be a square matrix over 'R', of order 'n' then a scalar number 'λ' is called Eigen Value of A, if there exists a non-zero column vector $v \in R^n$ such that

$$\boxed{Av = \lambda v}$$

Here 'v' is called Eigen Vector of A corresponding to Eigen Value 'λ'

I is Identity Matrix

Now $Av = \lambda v \Rightarrow Av = \lambda I v$

$\Rightarrow (A - \lambda I)v = 0$

Since $v \neq 0$ (given)

So $A - \lambda I$ is singular Matrix by theorem of homogenous system of eqs

$\therefore$ $\boxed{|A - \lambda I| = 0}$ ——— (1)

$|A - \lambda I|$ is called characteristic polynomial of matrix A

Every Root of eq (1) i.e value of λ is called Eigen Value of A or Characteristic Value of A

The value of 'V' corresponding to value of 'λ' given by $(A - \lambda I)v = 0$ is called Eigen Vector corresponding to λ.

(standard)

Example 15 Find the eigen values and corresponding eigenvectors of the matrix.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Sol

$$A - \lambda I = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \quad \text{——— (1)}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(3-\lambda)(2-\lambda) - 2\right] - 2\left[(2-\lambda) - 1\right] + 1\left[2 - (3-\lambda)\right] = 0$$

$$\Rightarrow (2-\lambda)\left[6 - 5\lambda + \lambda^2 - 2\right] - 2(1-\lambda) + (-1+\lambda) = 0$$

$$\Rightarrow 12 - 10\lambda + 2\lambda^2 - 4 - 6\lambda + 5\lambda^2 - \lambda^3 + 2\lambda - 2 + 2\lambda - 1 + \lambda = 0$$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$\Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$\therefore \lambda = 5, 1, 1$ (Eigen Values)

$$\begin{array}{c|cccc} 1 & 1 & -7 & 11 & -5 \\ & \downarrow & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & 0 \end{array}$$

$\lambda^2 - 6\lambda + 5 = 0$

$\lambda = \dfrac{6 \pm \sqrt{36 - 4 \cdot 1 \cdot 5}}{2}$

$= \dfrac{6 \pm 4}{2} = 5, 1$

Let Eigen Vector 'V' for $\lambda = 5$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$(A - \lambda I)V = 0$

put $\lambda = 5$ in (1)

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x + 2y + z = 0$
$x - 2y + z = 0$
$x + 2y - 3z = 0$

Homogeneous Eqs i.e. $AX = 0$

System of homogeneous 'm' eqs in 'n' unknowns has non trivial sol if Rank $A^* < n$

for Solution of Homogeneous Eqs Reduce in Echelon form

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & 1 \\ -3 & 2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$ by R_{12}

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -4 & 4 \\ 0 & 4 & -4 \end{bmatrix}$ by $3R_1 + R_2$ and $-R_1 + R_3$

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ by $-\frac{1}{4}R_2$, by $\frac{1}{4}R_3$

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $-R_2 + R_3$ (Echelon form)

$A^* = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Rank of $(A^*) = 2 <$ No. of unknowns $= 3$ $\therefore$ System has non-trivial sol.

$x - 2y + z = 0$ ——— (II)

$y - z = 0 \Rightarrow y = z$ — (III) Put in (II)

from (II) $x - 2y + y = 0$

$x - y = 0 \Rightarrow x = y$

(Giving Arbitrary Values) let $x = y = z = a$

then $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} =$

Eigen Vector 'V' corresponding to $\lambda = 5$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = [1 \; 1 \; 1]^t$ where $a = 1$ is arbitrary

Now For Eigen Value $\lambda=1$, Eigen vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(Scalar multiples of $[1\,1\,1]^t$ are all eigen vectors corresponding to $\lambda=5$ and they form a subspace of R^3 generated by $(1\,1\,1)^t$ which is also a basis of subspace (called the Eigen space of A corresponding to $\lambda=5$))

$$(A-\lambda I)v = 0$$

Put $\lambda=1$ in (1)

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce in Echelon form

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{R} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ by } -R_1+R_2,\ -R_1+R_3$$ Echelon form

Now

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x + 2y + z = 0$

Let $y = a$ & $z = b$ (giving arbitrary values)

So $x + 2a + b = 0 \Rightarrow x = -2a - b$

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2a-b \\ a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} -2a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} -b \\ 0 \\ b \end{bmatrix}$$

$$= a\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

linearly independent $\because \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is Echelon form.

Here we have two linearly independent vectors $[-2\ 1\ 0]^t$ and $[-1\ 0\ 1]^t$ corresponding to eigen vector $\lambda = 1$.

Any linear combination of $[-2\ 1\ 0]^t$ and $[-1\ 0\ 1]^t$ is also an Eigen vector corresponding to $\lambda = 1$.

The Set of linear combinations of these eigen vectors form a subspace of R^3. This subspace is called Eigenspace of A corresponding to $\lambda=1$

The basis of eigen space $= \{[-2\ 1\ 0]^t, [-1\ 0\ 1]^t\}$

Theorem

Non-zero eigen vectors of a matrix A corresponding to distinct eigen values are linearly independent.

Proof Let $v_1, v_2, \ldots v_n$ be non zero eigen vectors of matrix A corresponding to distinct eigen values $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ respectively.

We prove the theorem by induction

For $n=1$

then v_1 is linearly independent as $v_1 \neq 0$ so $a=0$.

For $n=K$ suppose the theorem is true i.e vectors $v_1, v_2 \ldots v_K$ are linearly independent. i.e $a_1v_1 + a_2v_2 + \ldots a_K v_K = 0$ —— ①

For $n=K+1$ Consider $a_1v_1 + a_2v_2 + \ldots a_K v_K + a_{K+1} v_{K+1} = 0$ —— ②

$$A(a_1v_1 + a_2v_2 + \ldots a_n v_K + a_{K+1} v_{K+1}) = A.0$$

$$\Rightarrow a_1 Av_1 + a_2 Av_2 + \ldots a_K Av_K + a_{K+1} Av_{K+1} = 0$$

$$\Rightarrow a_1 \lambda v_1 + a_2 \lambda v_2 + \ldots a_K \lambda_K v_K + a_{K+1} \lambda_{K+1} v_{K+1} = 0 \quad \text{—— ③} \quad \because Av = \lambda v \text{ By def.}$$

Multiply ② by λ_{K+1} $\Rightarrow a_1 \lambda_{K+1} v_1 + a_2 \lambda_{K+1} v_2 + \ldots a_K \lambda_{K+1} v_K + a_{K+1} \lambda_{K+1} v_{K+1} = 0$ —— ④

Subtracting ④ from ③ $\Rightarrow a_1(\lambda_1 - \lambda_{K+1})v_1 + a_2(\lambda_2 - \lambda_{K+1})v_2 + \ldots a_K(\lambda_K - \lambda_{K+1})v_K + 0 = 0$

From ① the vectors $v_1, v_2 \ldots v_K$ are linearly independent (As supposed)

So $a_1(\lambda_1 - \lambda_{K+1}) = 0$

$a_2(\lambda_2 - \lambda_{K+1}) = 0$

$\vdots$

$a_K(\lambda_K - \lambda_{K+1}) = 0$

Since $\lambda_1, \lambda_2 \ldots \lambda_K$ are all distinct So $\lambda_i - \lambda_{K+1} \neq 0$ $i=1,2 \ldots K$

$\Rightarrow a_i = 0$ $i = 1, 2 \ldots K$

using $a_i = 0$, $i = 1,2 \ldots K$ in ② $\Rightarrow a_{K+1} v_{K+1} = 0$

$\Rightarrow a_{K+1} = 0$ $\because v_{K+1}$ is nonzero

Thus vectors $v_1, v_2 \ldots v_{K+1}$ are all linearly independent

Hence by Induction $v_1, v_2, \ldots v_n$ are linearly independent.

Theorem Any two eigen vectors corresponding to two distinct eigen values of an orthogonal matrix are orthogonal.

Proof Let A be an $n\times n$ orthogonal matrix, having eigen vectors v_1 & v_2 corresponding to distinct eigen values λ_1 & λ_2.

Then by def $Av_1 = \lambda_1 v_1$ ——①

and $Av_2 = \lambda_2 v_2$ $\lambda_1 \neq \lambda_2$ $\because$ distinct eigenvalues (given)

taking transpose $(Av_2)^t = (\lambda_2 v_2)^t$

$v_2^t A^t = \lambda_2 v_2^t$ ——②

$\because Av_1 = \lambda_1 v_1$
$\times$ by $v_2^t A^t$
$\therefore Av_1 v_2^t A^t = \lambda v_1 v_2^t A^t$
$= \lambda_1 v_1 \lambda_2 v_2$
using ②

From ① & ② $(v_2^t A^t) A v_1 = (\lambda_2 v_2^t)(\lambda_1 v_1)$

$$v_2^t (A^t A) v_1 = \lambda_1 \lambda_2 v_2^t v_1$$

$$v_2^t I v_1 = \lambda_1 \lambda_2 v_2^t v_1$$

$$v_2^t v_1 = \lambda_1 \lambda_2 v_2^t v_1$$

$$v_2^t v_1 (1 - \lambda_1 \lambda_2) = 0 \quad \text{——③}$$

$$v_2^t v_1 (\lambda_2^2 - \lambda_1 \lambda_2) = 0$$

Since by theorem if λ is an eigen value of an orthogonal matrix then $|\lambda| = 1$ i,e $\lambda^2 = 1$ $\therefore \lambda_2^2 = 1$

$$v_2^t v_1 \lambda_2 (\lambda_2 - \lambda_1) = 0$$

As $\lambda_2 \neq \lambda_1$ & $\lambda_2 \neq 0$ $\because |\lambda_2| = 1$

$\therefore v_2^t v_1 = 0$

$\Rightarrow \langle v_2, v_1 \rangle = 0$

$\because$ (see Example 2) I.P of v_1 & v_2 col vector is $\langle v_1 \cdot v_2 \rangle = v_1^t v_2$

Hence v_1, v_2 are orthogonal vectors.

Theorem If λ is an eigen value of an orthogonal matrix, then $|\lambda|=1$

Proof Let A be a square matrix of order 'n'. Also A is orthogonal matrix and 'λ' be an eigen value of A. Then there exists a non-zero column vector $v \in R^n$ such that

$$AV = \lambda V \quad \text{--- (1)}$$

taking transpose

$$(AV)^t = (\lambda V)^t$$

$$V^t A^t = \lambda V^t \quad \text{--- (2)} \qquad \because \lambda \text{ is scalar.}$$

From (1) & (2)

$$(V^t A^t) AV = (\lambda V^t)\lambda V$$

($\times$ both sides of (2) by AV, then put $AV = \lambda V$ on R.H.S.

$$V^t (A^t A) V = \lambda^2 V^t V$$

$$V^t I V = \lambda^2 V^t V$$

$\because A$ is orthogonal so $A^t A = I$

$$V^t V = \lambda^2 V^t V$$

$$V^t V (1-\lambda^2) = 0$$

$$1-\lambda^2 = 0 \qquad \because V \neq 0 \text{ so } V^t V \neq 0$$

$$1 = \lambda^2$$

$$1 = |\lambda|^2$$

$$1 = |\lambda| \quad \text{Proved}$$