Ex 7.2

Q1 If A is an orthogonal matrix show that $A = 1$ or -1

Sol Since A is orthogonal Matrix

$\therefore \quad AA^t = I$

$\det(AA^t) = \det I$

By Product Theorem $\Rightarrow \det(A)\cdot\det(A^t) = 1 \qquad \because (\det(AB) = \det A \cdot \det B)$

$\Rightarrow \det(A)\cdot\det(A) = 1 \qquad \because \det(A^t) = \det(A)$

$\Rightarrow [\det(A)]^2 = 1$

$\det(A) = \pm 1$ Proved

2nd Method

Since A is orthogonal Matrix

$\therefore \quad A^t = A^{-1}$

$\det(A^t) = \det(A^{-1})$

$\det A = \dfrac{1}{\det(A)} \qquad \because \det(A^t) = \det A.$

$[\det A]^2 = 1$

$\det A = \pm 1$ Proved.

Prove that $\det A^{-1} = \dfrac{1}{\det A}$

We know $A^{-1}A = I$

$\det(A^{-1}A) = \det I$

By Product Theorem $\det A^{-1} \det A = 1$

$\det A^{-1} = \dfrac{1}{\det A}$

proved.

Q2 Find an orthogonal matrix whose first row is $\left(\frac{1}{\sqrt5}, \frac{2}{\sqrt5}\right)$

Sol Let $R_1 = \left(\frac{1}{\sqrt5}, \frac{2}{\sqrt5}\right)$ (given)

A vector orthogonal to R_1 is $R_2^\star = \begin{vmatrix} e_1 & e_2 \\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{vmatrix}$

$= \frac{2}{\sqrt5}e_1 - \frac{e_2}{\sqrt5}$

$= \left(\frac{2}{\sqrt5}, -\frac{1}{\sqrt5}\right)$

$\|R_2^\star\| = \sqrt{\frac{4}{5} + \frac{1}{5}} = \sqrt{\frac{5}{5}} = 1$

$R_2 = \dfrac{R_2^\star}{\|R_2^\star\|} = \dfrac{\left(\frac{2}{\sqrt5}, -\frac{1}{\sqrt5}\right)}{1} = \left(\frac{2}{\sqrt5}, -\frac{1}{\sqrt5}\right)$

$\therefore \quad A = \begin{bmatrix} \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} \end{bmatrix}$

To Check $AA^t = I$

$AA^t = \begin{bmatrix} \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} \end{bmatrix} = \begin{bmatrix} \frac{1}{5}+\frac{4}{5} & \frac{2}{5}-\frac{2}{5} \\ \frac{2}{5}-\frac{2}{5} & \frac{4}{5}+\frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ proved

Q3 Find an orthogonal matrix whose first row is multiple of (1,1,1)

Sol $w_1 = (1,1,1)$ given

$\|w_1\| = \sqrt{3}$

Normalised first Row $V_1 = \dfrac{w_1}{\|w_1\|} = \dfrac{(1,1,1)}{\sqrt{3}} = \left(\frac{1}{3}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

we can not use
$\begin{vmatrix} e_1 & e_2 \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \end{vmatrix} + \begin{vmatrix} e_2 & e_3 \\ -\frac{1}{\sqrt3} & \frac{1}{\sqrt3} \end{vmatrix} + \begin{vmatrix} e_3 & e_1 \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \end{vmatrix}$
because it becomes 0. and when it becomes 0 it is not applicable.

Let w_2 is orthogonal to V_1, where $w_2 = (x, y, z)$

$\therefore \langle V_1, w_2 \rangle = 0$

$\Rightarrow \dfrac{x}{\sqrt3} + \dfrac{y}{\sqrt3} + \dfrac{z}{\sqrt3} = 0$

$\Rightarrow x + y + z = 0$

Put $x = 0$

$\Rightarrow y + z = 0 \qquad \Rightarrow z = -y$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ -y \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$\therefore w_2 = (0, 1, -1)$

$\|w_2\| = \sqrt{0^2 + 1^2 + (-1)^2} = \sqrt{2}$

$V_2 = \dfrac{w_2}{\|w_2\|} = \dfrac{(0,1,-1)}{\sqrt2} = \left(0, \frac{1}{\sqrt2}, -\frac{1}{\sqrt2}\right)$

A vector orthogonal to V_1 & V_2 is

$$w_3 = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \\ 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{vmatrix}$$

$$= e_1\left(-\frac{1}{\sqrt6} - \frac{1}{\sqrt6}\right) - e_2\left(\frac{-1}{\sqrt6}\right) + e_3\left(\frac{1}{\sqrt6}\right)$$

$$w_3 = \left(-\frac{2}{\sqrt6}, \frac{1}{\sqrt6}, \frac{1}{\sqrt6}\right)$$

$$\|w_3\| = \sqrt{\frac{4}{6} + \frac{1}{6} + \frac{1}{6}} = 1$$

$$V_3 = \frac{w_3}{\|w_3\|} = \left(-\frac{2}{\sqrt6}, \frac{1}{\sqrt3}, \frac{1}{\sqrt6}\right)$$

Hence the required orthogonal Matrix is
$$\begin{pmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \\ 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \\ -\frac{2}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} \end{pmatrix}$$

Q4 Find an orthogonal matrix whose first row is $\left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

Let $R_1 = \boxed{\left[0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right]}$

A vector (row) orthogonal to R_1 is $R_2^{\star} = \begin{vmatrix} e_1 & e_2 \\ 0 & \frac{1}{\sqrt{5}} \end{vmatrix} + \begin{vmatrix} e_2 & e_3 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{vmatrix} + \begin{vmatrix} e_3 & e_1 \\ \frac{2}{\sqrt{5}} & 0 \end{vmatrix}$

$$= e_1\left(\frac{1}{\sqrt{5}}\right) + \frac{2}{\sqrt{5}}e_2 - \frac{1}{\sqrt{5}}e_3 + \left(0 - \frac{2}{\sqrt{5}}e_1\right)$$

$$= e_1\left(\frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}}\right) + \frac{2}{\sqrt{5}}e_2 - \frac{1}{\sqrt{5}}e_3$$

$$R_2^{\star} = -\frac{1}{\sqrt{5}}e_1 + \frac{2}{\sqrt{5}}e_2 - \frac{1}{\sqrt{5}}e_3 = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$$

$$\|R_2^{\star}\| = \sqrt{\frac{1}{5} + \frac{4}{5} + \frac{1}{5}} = \sqrt{\frac{6}{5}}$$

$$R_2 = \frac{R_2^{\star}}{\|R_2^{\star}\|} = \frac{\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)}{\sqrt{\frac{6}{5}}} = \frac{\sqrt{5}}{\sqrt{6}}\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) = \boxed{\left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)}$$

A vector orthogonal to R_1 & R_2 is $R_3^{\star} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{vmatrix}$

$$= e_1\left(\frac{-1}{\sqrt{30}} - \frac{4}{\sqrt{30}}\right) - e_2\left(\frac{2}{\sqrt{30}}\right) - e_3\left(+\frac{1}{\sqrt{30}}\right)$$

$$R_3^{\star} = \left(\frac{-5}{\sqrt{30}}, \frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right)$$

$$\|R_3^{\star}\| = \sqrt{\frac{25}{30} + \frac{4}{30} + \frac{1}{30}} = 1$$

$$R_3 = \frac{R_3^{\star}}{\|R_3^{\star}\|} = \boxed{\left(\frac{-5}{\sqrt{30}}, \frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right)}$$

Hence orthogonal Matrix $A = \begin{bmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ -\frac{5}{\sqrt{30}} & -\frac{2}{\sqrt{30}} & \frac{1}{\sqrt{30}} \end{bmatrix}$

Q5 Show that the products and inverses of orthogonal matrices are orthogonal. Hence show that orthogonal matrices form a group under multiplication.

Sol Let A & B be orthogonal matrices. So $A^tA = I \Rightarrow A^t = A^{-1}$
$B^tB = I \Rightarrow B^t = B^{-1}$

Now To Prove Product of orthogonal Matrix is Orthogonal i.e (AB) is orthogonal
we show $(AB)^t(AB) = I$

Therefore
$$(AB)^t(AB) = B^tA^tAB$$
$$= B^t(A^tA)B$$
$$= B^t I B$$
$$= B^tB$$
$$(AB)^t(AB) = I$$

So AB is an orthogonal Matrix.

Now To Prove Inverse of orthogonal Matrix is orthogonal. i.e A^{-1} is orthogonal
we show $(A^{-1})^t(A^{-1}) = I$

Therefore
$$(A^{-1})^t(A^{-1}) = (A^{-1})^t(A^t) \quad (\because (A^{-1}) = A^t \text{ since A is orthogonal})$$
$$= (AA^{-1})^t \quad \because (AB)^t = B^tA^t$$
$$= (I)^t$$
$$(A^{-1})^t(A^{-1}) = I$$

Hence A^{-1} is orthogonal.

To Prove Set of orthogonal matrices form a group under multiplication.

Let G = Set of orthogonal matrices.

i) Let $A, B \in G$ then $AB \in G$ $\because$ product of orthogonal is orthogonal
Hence G is closed under B O of multiplication.

ii) In Matrices Associative law holds.

(iii) Identity Matrix is always orthogonal Matrix so $I \in G$.

(iv) Inverse of each orthogonal matrix is orthogonal (as solved above)
So Inverse of each orthogonal matrix belonging to G is in G.
So Inverse exists in G.

All conditions are satisfied, Hence G is a group under multiplication