Ex 7.3

Q1 For each of the following matrices, find the characteristic polynomial, all eigen values and a basis of each eigen space.

(i) $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}$

Characteristic Polynomial is $|A-\lambda I|$

$$A - \lambda I = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{pmatrix} \quad \text{——— (1)}$$

$|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0 \qquad \because |A-\lambda I| = 0$$

$$\Rightarrow (3-\lambda)\{(4-\lambda)(3-\lambda)-2\} - \{6-2\lambda-2\} + \{2-4+\lambda\} = 0$$

$$\Rightarrow (3-\lambda)\{12-7\lambda+\lambda^2-2\} - 4 + 2\lambda + 2 - 4 + \lambda = 0$$

$$\Rightarrow (3-\lambda)(10-7\lambda+\lambda^2) - 6 + 3\lambda = 0$$

$$\Rightarrow 30 - 21\lambda + 3\lambda^2 - 10\lambda + 7\lambda^2 - \lambda^3 - 6 + 3\lambda = 0$$

$$\Rightarrow -\lambda^3 + 10\lambda^2 - 28\lambda + 24 = 0$$

$$|A-\lambda I| = \boxed{\lambda^3 - 10\lambda^2 + 28\lambda - 24}$$ Characteristic Polynomial.

Eigen Values (i.e Roots) are $\lambda = 2, 2, 6$

$$\begin{array}{r|rrrr} 1 & 1 & -10 & 28 & -24 \\ & \downarrow & 1 & -9 & 19 \\ \hline & 1 & -9 & 19 & -5 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & -10 & 28 & -24 \\ & \downarrow & 2 & -16 & 24 \\ \hline & 1 & -8 & 12 & 0 \end{array}$$

$$\lambda^2 - 8\lambda + 12 = 0$$

$$\lambda = \frac{8 \pm \sqrt{64-48}}{2}$$

$$\lambda = \frac{8 \pm 4}{2} = 2, 6$$

Let Eigen Vector 'V' for $\lambda = 2$ is $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$(A-\lambda I)V = 0$$

Put $\lambda = 2$ in (1)

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now reduce matrix in Echelon form

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \overset{R}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{by } -2R_1+R_2, \; -R_1+R_3$$

Now $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x + y + z = 0$

For sol. Giving arbitrary values to y & z, for Let $y = a$, $z = b$

$x + a + b = 0$

$x = -a - b$

then $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -a-b \\ a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} -b \\ 0 \\ b \end{bmatrix}$

$= +a\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

The Eigen vectors corresponding to $\lambda = 2$ are $[-1 \ 1 \ 0]^t$, $[-1 \ 0 \ 1]^t$

Basis of EigenSpace corresponding to $\lambda = 2$ is $\{[-1 \ 1 \ 0]^t, [-1 \ 0 \ 1]^t\}$

Now for $\lambda = 6$ Eigen Vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$(A - \lambda I)v = 0$

Put $\lambda = 6$ in (1) $\begin{bmatrix} -3 & 1 & 1 \\ 2 & -2 & 2 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now reduce Matrix in Echelon form $\begin{bmatrix} -3 & 1 & 1 \\ 2 & -2 & 2 \\ 1 & 1 & -3 \end{bmatrix} \overset{R}{\sim} \begin{bmatrix} 1 & 1 & -3 \\ 2 & -2 & 2 \\ -3 & 1 & 1 \end{bmatrix}_{\text{by } R_{13}} \overset{R}{\sim} \begin{bmatrix} 1 & 1 & -3 \\ 0 & -4 & 8 \\ 0 & 4 & -8 \end{bmatrix}_{\text{by } -2R_1+R_2,\ 3R_1+R_3} \overset{R}{\sim} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & -2 \\ 0 & 4 & -8 \end{bmatrix}_{\text{by } \frac{-1}{4}R_2}$

$\overset{R}{\sim} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}_{\text{by } -4R_2+R_3}$

$x + y - 3z = 0$

$y - 2z = 0 \Rightarrow y = 2z$

Let $z = a$ be arbitrary value

$x + 2a - 3a = 0 \Rightarrow x = a$, $y = 2a$, $z = a$

$\therefore v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ 2a \\ a \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$v = [1 \ 2 \ 1]^t$

Eigen Vector corresponding to $\lambda = 6$ is $[1 \ 2 \ 1]^t$

Basis of EigenSpace corresponding to $\lambda = 6$ is $\{[1 \ 2 \ 1]^t\}$

x————————x

Q 1 (ii)
$$A=\begin{pmatrix}1&2&2\\1&2&-1\\-1&1&4\end{pmatrix}$$

$$A-\lambda I=\begin{pmatrix}1&2&2\\1&2&-1\\-1&1&4\end{pmatrix}-\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$A-\lambda I=\begin{pmatrix}1-\lambda&2&2\\1&2-\lambda&-1\\-1&1&4-\lambda\end{pmatrix} \quad \text{———} \quad (1)$$

$$|A-\lambda I|=0$$

$$\Rightarrow \begin{vmatrix}1-\lambda&2&2\\1&2-\lambda&-1\\-1&1&4-\lambda\end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)(4-\lambda)+1\right]-2\left[(4-\lambda)-1\right]+2\left[1+(2-\lambda)\right]=0$$

$$\Rightarrow (1-\lambda)\left[8-6\lambda+\lambda^2+1\right]-2\left[\cancel{3-\lambda}\right]+2\left[\cancel{3-\lambda}\right]=0$$

$$\Rightarrow (1-\lambda)(\lambda^2-6\lambda+9)=0$$

$$\Rightarrow \lambda^2-6\lambda+9-\lambda^3+6\lambda^2-9\lambda=0$$

$$\Rightarrow -\lambda^3+7\lambda^2-15\lambda+9=0$$

$$\Rightarrow \lambda^3-7\lambda^2+15\lambda-9=0$$

$$|A-\lambda I|=\boxed{\lambda^3-7\lambda^2+15\lambda-9}$$ Characteristic Polynomial

Eigen Values (i.e Roots) are $\lambda=1,3,3$

$$\begin{array}{r|rrrr} 1 & 1 & -7 & 15 & -9 \\ & \downarrow & 1 & -6 & 9 \\ \hline & 1 & -6 & 9 & 0 \end{array}$$

$$\lambda^2-6\lambda+9=0$$
$$\lambda=\frac{6\pm\sqrt{36-36}}{2}$$
$$\lambda=\frac{6\pm0}{2}=3,3$$

For $\lambda=1$

Let Eigen Vector 'V' $=\begin{bmatrix}x\\y\\z\end{bmatrix}$

$$(A-\lambda I)V=0$$

Put $\lambda=1$ in (1)
$$\begin{pmatrix}0&2&2\\1&1&-1\\-1&1&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Now reduce matrix in Echelon form
$$\begin{bmatrix}0&2&2\\1&1&-1\\-1&1&3\end{bmatrix} \overset{R}{\sim} \begin{bmatrix}1&1&-1\\0&2&2\\-1&1&3\end{bmatrix}_{\text{by } R_{12}} \overset{R}{\sim} \begin{bmatrix}1&1&-1\\0&2&2\\0&2&2\end{bmatrix}_{\text{by } R_1+R_3}$$

$$\overset{R}{\sim} \begin{bmatrix}1&1&-1\\0&1&1\\0&2&2\end{bmatrix}_{\text{by } \frac{1}{2}R_2} \overset{R}{\sim} \begin{bmatrix}1&1&-1\\0&1&1\\0&0&0\end{bmatrix}_{\text{by } -2R_2+R_3}$$ Echelon form

Now $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x + y - z = 0$

$y + z = 0 \quad \Rightarrow y = -z$

Let $z = a$ be arbitrary value

$x + (-a) - a = 0 \quad \Rightarrow x = 2a$

$y = -a$

$z = a$

$\therefore v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2a \\ -a \\ a \end{bmatrix} = a\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

<u>Eigen Vector 'v'</u> $= [2 \;\; -1 \;\; 1]^t$ corresponding to $\lambda = 1$

<u>Basis of Eigen Space</u> corresponding to Eigen Value $\lambda = 1$ is $\{[2 \;\; -1 \;\; 1]^t\}$

Now for $\lambda = 3$ Eigen Vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$(A - \lambda I)v = 0$

Put $\lambda = 3$ in (1) $\begin{bmatrix} -2 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now Reduce Matrix in Echelon form $\begin{bmatrix} -2 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \overset{R}{\sim} \begin{bmatrix} 1 & -1 & -1 \\ -2 & 2 & 2 \\ -1 & 1 & 1 \end{bmatrix}_{\text{by } R_{12}} \overset{R}{\sim} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\text{by } 2R_1 + R_2,\; R_1 + R_3}$

Now $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x - y - z = 0$

Let $y = a$, $z = b$ be arbitrary values.

$\therefore \Rightarrow x = a + b.$

Hence $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a+b \\ a \\ b \end{bmatrix} = \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} b \\ 0 \\ b \end{bmatrix}$

$= a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Therefore the <u>Eigen Vectors</u> corresponding to <u>Eigen Value $\lambda = 3$</u> are $[1 \;\; 1 \;\; 0]^t$ & $[1 \;\; 0 \;\; 1]^t$

<u>Basis of Eigen Space</u> corresponding to $\lambda = 3$ is $\{[1 \;\; 1 \;\; 0]^t, [1 \;\; 0 \;\; 1]^t\}$

Q1(iii) $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$

$$A-\lambda I = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{pmatrix} \quad \text{———} (1)$$

$|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(4-\lambda)(-5-\lambda)+18\right] + 3\left[3(4-\lambda)-18\right] + 3\left[-18+6(5+\lambda)\right] = 0$$

$$\Rightarrow (1-\lambda)\left[-20+\lambda+\lambda^2+18\right] + 3\left[12-3\lambda-18\right] + 3\left[12+6\lambda\right] = 0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2+\lambda-2\right] + 3\left[-6-3\lambda\right] + (36+18\lambda) = 0$$

$$\Rightarrow \cancel{\lambda^2}+\lambda-2-\lambda^3-\cancel{\lambda^2}+2\lambda-18-9\lambda+36+18\lambda = 0$$

$$\Rightarrow -\lambda^3+12\lambda+16 = 0$$

$|A-\lambda I| = \boxed{\lambda^3-12\lambda-16}$ Characteristic Polynomial

Eigen Values (i.e) Roots are $\lambda = 4, -, -2$

4	1	0	-12	-16
	↓	4	16	16
	1	4	4	(0

$\lambda^2 + 4\lambda + 4 = 0$

$\lambda = \dfrac{-4 \pm \sqrt{16-16}}{2}$

$= \dfrac{-4 \pm 0}{2} = -2, -2$

For $\lambda = 4$ Let Eigen Vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$(A-\lambda I)v = 0$

Put $\lambda = 4$ in (1)
$$\begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now reduce Matrix in Echelon Form
$$\begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \overset{R}{\sim} \begin{pmatrix} 1 & 1 & -1 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \text{by } -\tfrac{1}{3}R_1 \quad \overset{R}{\sim} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -12 & 6 \\ 0 & -12 & 6 \end{pmatrix} \text{by } \begin{matrix} -3R_1+R_2 \\ -6R_1+R_3 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 1 & -1\\ 0 & 1 & -\frac{1}{2}\\ 0 & -12 & 6\end{bmatrix}_{-\frac{1}{12}R_2}$$

Now $\begin{bmatrix}1 & 1 & -1\\ 0 & 1 & -\frac{1}{2}\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$\Rightarrow$ $x+y-z=0$

$y-\frac{z}{2}=0 \Rightarrow y=\frac{z}{2}$

Let $z=a$ be arbitrary value

$\therefore y=\frac{a}{2}$

and $x+\frac{a}{2}-a=0 \Rightarrow x=a-\frac{a}{2}$

$x=\frac{a}{2}$

$\therefore v=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a/2\\ a/2\\ a\end{bmatrix}=a\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ 1\end{bmatrix}$

Note if we take $z=2a$
then $y=a$
and $x+a-2a=0$
$x=a$
$\therefore v=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ a\\ 2a\end{bmatrix}$
$=a\begin{bmatrix}1\\ 1\\ 2\end{bmatrix}$ Ans in Book

Eigen Vector $v=\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 1\end{bmatrix}^t$ corresponding to $\lambda=4$

Basis of Eigen Space corresponding to Eigen Value $\lambda=4$ is $\left\{\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 1\end{bmatrix}^t\right\}$

Now for $\lambda=-2$ Eigen Vector $v=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$

$(A-\lambda I)v=0$

Put $\lambda=-2$ in ① $\begin{bmatrix}3 & -3 & 3\\ 3 & -3 & 3\\ 6 & -6 & 6\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

Now reduce Matrix in Echelon Form $\begin{bmatrix}3 & -3 & 3\\ 3 & -3 & 3\\ 6 & -6 & 6\end{bmatrix}\overset{R}{\sim}\begin{bmatrix}1 & -1 & 1\\ 3 & -3 & 3\\ 6 & -6 & 6\end{bmatrix}_{\frac{1}{3}R_1}\overset{R}{\sim}\begin{bmatrix}1 & -1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}_{\substack{-3R_1+R_2\\ -6R_1+R_3}}$ Echelon form

Now $\begin{bmatrix}+1 & -1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$\Rightarrow$ $x-y+z=0$

Let $y=a$, $z=b$ be arbitrary values

$\therefore$ $x-a+b=0 \Rightarrow x=a-b$

Hence $v=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a-b\\ a\\ b\end{bmatrix}=\begin{bmatrix}a\\ a\\ 0\end{bmatrix}+\begin{bmatrix}-b\\ 0\\ b\end{bmatrix}$

$=a\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}+b\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$

Therefore the Eigen Vectors corresponding to Eigen Value $\lambda=-2$ are $\begin{bmatrix}1 & 1 & 0\end{bmatrix}^t$ and $\begin{bmatrix}-1 & 0 & 1\end{bmatrix}^t$

Basis of Eigen Space corresponding to $\lambda=-2$ is $\left\{\begin{bmatrix}1 & 1 & 0\end{bmatrix}^t, \begin{bmatrix}-1 & 0 & 1\end{bmatrix}^t\right\}$

Q1 (iv) $A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{pmatrix} \quad \text{——— (1)}$$

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-3-\lambda)\left[(5-\lambda)(-2-\lambda)+6\right] - 1\left[-7(-2-\lambda)-6\right] - 1\left[-42+6(5-\lambda)\right] = 0$$

$$\Rightarrow (-3-\lambda)\left[-10-5\lambda+2\lambda+\lambda^2+6\right] - 1\left[+14+7\lambda-6\right] - \left[-42+30-6\lambda\right] = 0$$

$$\Rightarrow (-3-\lambda)(-4-3\lambda+\lambda^2) - 8 - 7\lambda + 12 + 6\lambda = 0$$

$$\Rightarrow 12 + 9\lambda - \cancel{3\lambda^2} + 4\lambda + \cancel{3\lambda^2} - \lambda^3 + 4 - \lambda = 0$$

$$\Rightarrow -\lambda^3 + 12\lambda + 16 = 0$$

$|A - \lambda I| = \boxed{\lambda^3 - 12\lambda - 16}$ Charactristic Polynomial

Eigen Values are $\lambda = 4, -2, -2,$

$$\begin{array}{c|cccc} & 1 & 0 & -12 & -16 \\ 4 & \downarrow & 4 & 16 & 16 \\ \hline & 1 & 4 & 4 & 0 \end{array}$$

$\lambda^2 + 4\lambda + 4 = 0$

$\lambda = \dfrac{-4 \pm \sqrt{16-16}}{2}$

$= \dfrac{-4 \pm 0}{2} = -2, -2$

For $\lambda = 4$ Eigen Vectors $V = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$(A - \lambda I)V = 0$

Put $\lambda = 4$ in (1)
$$\begin{pmatrix} -7 & 1 & -1 \\ -7 & 1 & -1 \\ -6 & 6 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now reduce Matrix in Echelon form
$$\begin{pmatrix} -7 & 1 & -1 \\ -7 & 1 & -1 \\ -6 & 6 & -6 \end{pmatrix} \overset{R}{\sim} \begin{pmatrix} -6 & 6 & -6 \\ -7 & 1 & -1 \\ -7 & 1 & -1 \end{pmatrix} R_{13} \overset{R}{\sim} \begin{pmatrix} 1 & -1 & 1 \\ -7 & 1 & -1 \\ -7 & 1 & -1 \end{pmatrix} -\tfrac{1}{6}R_1$$

$$\overset{R}{\sim} \begin{pmatrix} 1 & -1 & 1 \\ 0 & -6 & 6 \\ 0 & -6 & 6 \end{pmatrix} \begin{matrix} 7R_1+R_2 \\ 7R_1+R_3 \end{matrix} \overset{R}{\sim} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & -6 & 6 \end{pmatrix} -\tfrac{1}{6}R_2 \overset{R}{\sim} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} 6R_2+R_3 \text{ Echelon}$$

Now $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x - y + z = 0$

$y - z = 0 \Rightarrow y = z$

Let $z = a$ be the arbitrary value of

$y = a$

$x - a + a = 0 \Rightarrow x = 0$

Hence $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$ Eigen vector corresponding to $\lambda = 4$ is $[1\ 1\ 1]^t$

Basis of Eigen Space corresponding to $\lambda = 4$ is $\{[1\ 1\ 1]^t\}$

For $\lambda = -2$ Eigen Vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$(A - \lambda I)v = 0$

Put $\lambda = -2$ in ①
$\begin{bmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now reduce Matrix in Echelon Form
$\begin{bmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{bmatrix} \overset{R}{\sim} \begin{bmatrix} 1 & -1 & 1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{bmatrix} -1R_1 \overset{R}{\sim} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 6 \\ 0 & 0 & 6 \end{bmatrix} \begin{matrix} \\ 7R_1+R_2 \\ 6R_1+R_3 \end{matrix}$

$\overset{R}{\sim} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 6 \end{bmatrix} \frac{1}{6}R_2 \overset{R}{\sim} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \text{Echelon form} \\ \\ -6R_2+R_3 \end{matrix}$

Now $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad x - y + z = 0$

$z = 0$

$\Rightarrow \quad x - y + 0 = 0 \Rightarrow x = y$

Let $y = a \Rightarrow x = a$

Hence $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Eigen vector corresponding to $\lambda = -2$ is $[1\ 1\ 0]^t$

Basis of Eigen Space corresponding to $\lambda = -2$ is $\{[1\ 1\ 0]^t$

x———————x

Imp Q2 Show that eigen values of a diagonal matrix are its diagonal elements and the eigen vectors are the standard basis vectors.

Sol Let $A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$ Diagonal Matrix

In diagonal matrix diagonal elements are non zero.

$$A-\lambda I = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} a_{11}-\lambda & 0 & 0 & \cdots & 0 \\ 0 & a_{22}-\lambda & 0 & \cdots & 0 \\ 0 & 0 & a_{33}-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}-\lambda \end{pmatrix} \quad (1)$$

$$|A-\lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & 0 & 0 & \cdots & 0 \\ 0 & a_{22}-\lambda & 0 & \cdots & 0 \\ 0 & 0 & a_{33}-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & a_{nn}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)\cdots\cdots(a_{nn}-\lambda) = 0$$

Since this is determinant of triangular matrix, so it is expanded by Product of diagonal elements

$$\Rightarrow (a_{11}-\lambda)=0, \ (a_{22}-\lambda)=0, \ \cdots\cdots (a_{nn}-\lambda)=0$$

$$\Rightarrow \lambda = a_{11}, a_{22}, a_{33} \cdots\cdots a_{nn}$$

i.e Eigen Values are diagonal elements Proved.

Now For $\lambda = a_{11}$ Eigen Vector $v_1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$(A-\lambda I)v_1 = 0$$

$$\Rightarrow \begin{bmatrix} a_{11}-a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22}-a_{11} & 0 & \cdots & 0 \\ 0 & 0 & a_{33}-a_{11} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & a_{nn}-a_{11} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\Rightarrow (a_{11}-a_{11})x_1 + 0\cdot x_2 + 0\cdot x_3 + \cdots 0\cdot x_n = 0 \quad \Rightarrow 0\cdot x_1 = 0 \quad \Rightarrow x_1$ is non-zero,

$0\cdot x_1 + (a_{22}-a_{11})x_2 + 0\cdot x_3 + \cdots 0\cdot x_n = 0 \quad \Rightarrow (a_{22}-a_{11})x_2 = 0 \quad \Rightarrow x_2 = 0$ $\because (a_{22}-a_{11}) \neq 0$ diagonal elements are non-zero

$0\cdot x_1 + 0\cdot x_2 + (a_{33}-a_{11})x_3 + \cdots 0\cdot x_n = 0 \quad \Rightarrow (a_{33}-a_{11})x_3 = 0 \quad \Rightarrow x_3 = 0$ $\because (a_{33}-a_{11})$ is diagonal element

$\vdots$

$0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + \cdots (a_{nn}-a_{11})x_n = 0 \quad \Rightarrow (a_{nn}-a_{11})x_n = 0 \quad \Rightarrow x_n = 0$ $\because (a_{nn}-a_{11}) \neq 0$ being diagonal element

Let $\boxed{x_1 = a}$ giving arbitrary Value

Hence $v_1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow v_1 = [1 \ 0 \ 0 \cdots 0]^t$

Similarly for $\lambda = a_{22}, a_{33}, \cdots a_{nn}$

we have $v_2 = [0 \ 1 \ 0 \cdots 0]^t$, $v_3 = [0 \ 0 \ 1 \cdots 0]^t, \cdots, v_n = [0 \ 0 \cdots 1]^t$

Hence the Eigen Vectors $v_1, v_2, v_3, \ldots v_n$ are Standard basis for R^n

Q3. Show that A and A^t have the same eigen values. Give an example where A and A^t have different eigen vectors (A is a square Matrix)

Sol Let A (Square Matrix) $= \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$

Eigen Values of A are given by $|A - \lambda I| = 0$

$$\therefore A - \lambda I = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ a_{31} & a_{32} & a_{33} & & a_{3n} \\ \vdots & \vdots & \vdots & & \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & 0 & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn}-\lambda \end{vmatrix} = 0 \qquad \text{—— (1)}$$

Now $A^t = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{n3} \\ \vdots & & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn}-\lambda \end{pmatrix}$

Eigen values of A^t are given by $|A^t - \lambda I| = 0$

$$|A^t - \lambda I| = 0 \Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22}-\lambda & a_{32} & \cdots & a_{n2} \\ a_{13} & a_{23} & a_{33}-\lambda & \cdots & a_{n3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \qquad \text{—— (11)}$$

We know that the value of determinant is unchanged if rows and columns are interchanged (i.e transpose)

$$\therefore \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} = \begin{vmatrix} a_{11}-\lambda & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22}-\lambda & a_{32} & \cdots & a_{n2} \\ \vdots & \vdots & \vdots & & \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

$$|A - \lambda I| = |A^t - \lambda I|$$

Hence Eigen Values of A & A^t are same

7-3-11

Example of different Eigen Vectors of A & A^t

For Eigen Value $|A-\lambda I|=0$
For Eigen Vector $(A-\lambda I)v=0$

Let $A=\begin{bmatrix}2 & 4\\ 1 & 5\end{bmatrix}$

$$A-\lambda I=\begin{bmatrix}2 & 4\\ 1 & 5\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2-\lambda & 4-0\\ 1-0 & 5-\lambda\end{bmatrix}$$

$$|A-\lambda I|=0 \Rightarrow \begin{vmatrix}2-\lambda & 4\\ 1 & 5-\lambda\end{vmatrix}=0$$

$$\Rightarrow (5-\lambda)(2-\lambda)-4=0$$

$$\Rightarrow 10-7\lambda+\lambda^2-4=0$$

$$\Rightarrow \lambda^2-7\lambda+6=0$$

$$\Rightarrow \lambda=\frac{7\pm\sqrt{49-24}}{2}=\frac{7\pm 5}{2}$$

$$\Rightarrow \lambda=6,1 \quad \text{Eigen Values}$$

For $\lambda=1$ Eigen Vector of A is given by $(A-\lambda I)v=0$

Put $\lambda=1$ in ①

$$\Rightarrow \begin{bmatrix}1 & 4\\ 1 & 4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$\Rightarrow x+4y=0 \quad \Rightarrow x=-4y \quad \Rightarrow x=-4a$ where a is arbitrary

$x+4y=0 \quad \Rightarrow -4a+4y=0 \Rightarrow y=\frac{4a}{4}=a$

$$\therefore v=\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-4a\\ a\end{bmatrix}=a\begin{bmatrix}-4\\ 1\end{bmatrix} \qquad \boxed{v=[-4 \quad 1]^t} \text{ --- ③}$$

Say $\boxed{a=1}$

For $\lambda=1$ the Eigen Vector of A^t is given by $(A^t-\lambda I)v=0$

$$A^t-\lambda I=\begin{bmatrix}2 & 1\\ 4 & 5\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2-\lambda & 1\\ 4 & 5-\lambda\end{bmatrix} \text{ --- ②}$$

$$|A^t-\lambda I|=0 \Rightarrow \begin{vmatrix}2-\lambda & 1\\ 4 & 5-\lambda\end{vmatrix}=(2-\lambda)(5-\lambda)-4 \Rightarrow \lambda=6,1 \text{ Eigen Values}$$

$(A^t-\lambda I)v=0$

Put $\lambda=1$ in ②

$$\Rightarrow \begin{bmatrix}1 & 1\\ 4 & 4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \quad \Rightarrow \begin{matrix}x+y=0\\ 4x+4y=0\end{matrix}$$

$\Rightarrow x=-y=-a$ arbitrary Value.

$$\therefore v=\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-a\\ a\end{bmatrix}=a\begin{bmatrix}-1\\ 1\end{bmatrix}$$

Say $\boxed{a=1}$

$$\therefore \boxed{v=[-1 \quad 1]^t} \text{ --- ④}$$

from ③ & ④ Eigen Vector for A & A^t are different.

(4) Suppose v is an eigen vector of a square matrix A corresponding to the eigen value λ. Show that for $n > 0$, v is also an eigen vector of A^n corresponding to λ^n.

Sol

(given) $AV = \lambda V$ —— (I)

∵ By def 'V' is an eigen vector of 'A' corresponding to eigen value λ

To Prove $A^nV = \lambda^n V$

∴ we have to show that V is the eigen vector of A^n corresponding to eigen value λ^n

We prove by induction

For $n=1$ $\quad A^1V = \lambda^1 V$ $\quad$ true by (I) $\quad$ C-1 is satisfied

For $n=K$ $\quad A^K V = \lambda^K V$ $\quad$ true supposed. (II)

Consider $A^{K+1}V = A(A^K V)$

$= A(\lambda^K V)$ $\quad$ using (II)

$= \lambda^K(AV)$ $\quad$ ∵ λ^K is scalar

$= \lambda^K(\lambda V)$ $\quad$ using (I)

$A^{K+1}V = \lambda^{K+1}V$ $\quad$ C-2 is satisfied

Hence the result is true for all +ve integral values of 'n'.

x———————x

(5) If λ is an eigen value of a non-singular matrix A, then λ^{-1} is an eigen value of A^{-1}.

Sol We know

(given) ∵ λ is an eigen value of A

A is non singular matrix implies A^{-1} exists.

Let v is eigen vector corresponding to eigen value λ.

$AV = \lambda V$

$A^{-1}(AV) = A^{-1}(\lambda V)$

$(A^{-1}A)V = \lambda(A^{-1}V)$

$IV = \lambda(A^{-1}V)$

$V = \lambda(A^{-1}V)$

$\lambda^{-1}V = A^{-1}V$ $\quad$ ∵ λ is scalar no.

Hence λ^{-1} is the eigen value of A^{-1}.

x———————x

⑥ If A & B are square matrices show that AB & BA have same eigen values.

Sol $|A-\lambda I|=0$ gives the Eigen values 'λ' of A.

$\therefore |AB-\lambda I|=0$ gives the Eigen values 'λ' of AB

$\Rightarrow |B^{-1}B(AB) - B^{-1}B(\lambda I)| = 0$ $\quad$ x by $B^{-1}B$

$\Rightarrow |B^{-1}(BA)B - \lambda(B^{-1}B)I| = 0$

$\Rightarrow |B^{-1}(BA-\lambda I)B| = 0$

$\Rightarrow |B^{-1}|\,|BA-\lambda I|\cdot|B| = 0$ (By Product Th)

$\Rightarrow \frac{1}{|B|}\,|BA-\lambda I|\,|B| = 0$ $\quad \because \det A^{-1} = \frac{1}{\det A}$

$\Rightarrow |BA-\lambda I| = 0$ gives Eigen Values 'λ' of BA.

⑥ 2nd Method

If λ be an eigen value of A, corresponding to eigen vector V then $AV = \lambda V$

Let λ be an eigen value of AB. then $(AB)V = \lambda V$

$B(AB)V = B(\lambda V)$ $\quad$ x by B

$(BA)(BV) = \lambda(BV)$

which shows that BV is an eigen vector of BA with eigenvalue λ. Thus eigen values of BA & AB are same

⑦ If $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigenvalues of a square matrix of order 'n' then $K\lambda_1, K\lambda_2, \ldots K\lambda_n$, where K is a scalar, are eigenvalues of KA.

Sol If λ be an eigen value of A corresponding to eigen vector V then $AV = \lambda V$

If $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigenvalues of A of order 'n' corresponding to eigen vector V then $AV = (\lambda_1, \lambda_2 \cdots \lambda_n)V$

$K A . V = K(\lambda_1, \lambda_2 \cdots \lambda_n)V$

$(KA)V = (K\lambda_1, K\lambda_2 \cdots K\lambda_n)V$

which shows $K\lambda_1, K\lambda_2 \ldots K\lambda_n$ are eigenvalues of KA.

⑧ Suppose v is an eigen vector of $n\times n$ matrices A & B. Show that v is also an eigen vector of $aA+bB$, where a & b are any scalar.

Sol v is the eigenvector of A & B corresponding to eigen value λ_1 & λ_2

then $Av = \lambda_1 v$ and $Bv = \lambda_2 v$

$\Rightarrow aAv = a\lambda_1 v$ —(i) $\quad bBv = b\lambda_2 v$ —(ii)

Adding (i) & (ii)

$$aAv + bBv = a\lambda_1 v + b\lambda_2 v$$

$$(aA+bB)v = (a\lambda_1 + b\lambda_2)v$$

$\Rightarrow$ v is the eigenvector of matrix $aA+bB$

x———x

⑨ Let λ be an eigen value of a square matrix A. Let V_λ denote the set of all eigen vectors of A corresponding to eigen value λ. Show that V_λ is a subspace of V (V_λ is the eigen space of A corresponding to λ)

Sol To Prove V_λ is subspace of V.

Let v_1, v_2 belong to V_λ, a_1, a_2 be scalars then

$$Av_1 = \lambda v_1 \quad \text{—(i)}$$
$$Av_2 = \lambda v_2 \quad \text{—(ii)}$$

and
$$A(a_1v_1 + a_2v_2) = a_1Av_1 + a_2Av_2$$
$$= a_1\lambda v_1 + a_2\lambda v_2$$
$$= \lambda(a_1v_1 + a_2v_2)$$

So $a_1v_1 + a_2v_2$ is an eigen vector of 'A' corresponding to eigen value 'λ'.

Thus $a_1v_1 + a_2v_2$ belongs to V_λ.

Hence V_λ is a subspace of V. (Since if $w_1, w_2 \in W$ and $a, b \in F$ imply $aw_1 + bw_2 \in W$ then W is a subspace of V. where $W \subseteq V$)

x———x

Q/10 For the linear transformation $T: R^2 \to R^2$, find all eigenvalues, eigenvectors and a basis for each eigen space.

(i) $T(x,y) = (3x+3y, x+5y)$

We first find matrix of given linear transformation w.r.t standard basis, $e_1 = (1,0)$, $e_2 = (0,1)$

Standard Basis for $R^2 = \{e_1 = (1,0), e_2 = (0,1)\}$

$T(e_1) = T(1,0) = (3\cdot1+3\cdot0, 1+5\cdot0)$

$= (3, 1) \qquad = 3e_1 + e_2$

$T(e_2) = T(0,1) = (3\cdot0+3\cdot1, 0+5\cdot1)$

$= (3, 5) \qquad = 3e_1 + 5e_2$

Matrix of linear transformation i.e 'T' is $\begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} = A$ (say)

$$A - \lambda I = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 3 \\ 1 & 5-\lambda \end{bmatrix} \quad \text{———} (1)$$

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 3-\lambda & 3 \\ 1 & 5-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)(5-\lambda) - 3 = 0$

$\Rightarrow 12 - 8\lambda + \lambda^2 = 0$

$\Rightarrow \lambda = 6, 2$ Eigen Values

$\lambda^2 - 8\lambda + 12 = 0$

$\lambda = \dfrac{8 \pm \sqrt{64 - 4.1.12}}{2}$

$= \dfrac{8 \pm \sqrt{16}}{2}$

$= 6, 2$

For $\lambda = 6$ let Eigen Vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$

$(A - \lambda I)v = 0 \Rightarrow \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

put $\lambda = 6$ in (1)

$\Rightarrow -3x + 3y = 0 \qquad \Rightarrow x = \frac{\cancel{3}}{\cancel{3}}y$

$x - y = 0$

Let $x = a$ any arbitrary value.

$\therefore v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix} = a\begin{bmatrix} \ \end{bmatrix}$

$\therefore$ Eigen Vector for $\lambda = 6$ is $[1 \quad 1]^t$

The set $\{[1 \quad 1]^t\}$ generates the subspace of R^2 known as EigenSpace of A.

Basis of Eigen Space $\{[1 \quad 1]^t\}$

For $\lambda = 2$

$(A - \lambda I)v = 0$

$\Rightarrow \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x + 3y = 0 \Rightarrow x = -3y$ Let $y = a$ any arbitrary value

$x + 3y = 0$

$v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3a \\ a \end{bmatrix} = a\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\therefore$ Eigen Vector for $\lambda = 2$ is $[-3 \quad 1]^t$

Any linear combination of $[-3 \quad 1]^t$ is also Eigen Vector for $\lambda = 2$. The set $\{[-3 \quad 1]^t\}$ i.e i.e set of linear combinations is a subspace of R^2 called EigenSpace of A having basis $\{[-3 \quad 1]^t\}$

10 (ii) For the linear transformation $T: R^2 \to R^2$ find all eigenvalues, eigen Vectors and Basis for eigen Space

$T(x,y) = (y,x)$

$T(e_1) = T(1,0) = (0,1) = 0\cdot e_1 + e_2$

$T(e_2) = T(0,1) = (1,0) = 1\cdot e_1 + 0\cdot e_2$

$e_1 = (1,0)$, $e_2 = (0,1)$ standard Basis for R^2

Matrix of T is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A$

i.e. Matrix of linear Transformation

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \quad \text{—①}$$

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

$\lambda^2 = 1 \Rightarrow \lambda = 1, -1$ Eigen Values.

For $\lambda = 1$

$(A - \lambda I)V = 0 \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Put $\lambda = 1$ in ①

$\Rightarrow$ $-x + y = 0$
$x - y = 0$ $\Rightarrow y = x$ Let $y = a$, arbitrary value! $\therefore x = a$

$\therefore V = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigen Vector corresponding to $\lambda = 1$ is $[1 \;\; 1]^t$

Any linear combination of $[1 \;\; 1]^t$ is also the eigenvector for $\lambda = 1$
The Set $\{[1 \;\; 1]^t\}$ i.e. set of linear combinations is a subspace of R^2 called Eigen Space having basis $\{[1 \;\; 1]^t\}$.

For $\lambda = -1$

$(A - \lambda I)V = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Put $\lambda = 1$ in ①

$\Rightarrow$ $x + y = 0$
$x + y = 0$ $\Rightarrow x = -y$ Let $y = a$ arbitrary value! $\therefore x = -a$

$\therefore V = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -a \\ a \end{bmatrix} = a\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Eigen Vector corresponding to $\lambda = -1$ is $[-1 \;\; 1]^t$

Any linear combination of $[-1 \;\; 1]^t$ is also eigen vector for $\lambda = -1$
The Set $\{[-1 \;\; 1]^t\}$ i.e. set of linear combinations is a subspace of R^2 called Eigen Space having basis $\{[-1 \;\; 1]^t\}$

Q10 (iii) For the linear transformation $T: R^2 \to R^2$ find all eigen values and a basis for each eigen space.

7.3-17

$T(x, y) = (y, -x)$

$T(e_1) = T(1,0) = (0,-1) := 0 e_1 + e_2$

$T(e_2) = T(0,1) = (1,0) := 1 \cdot e_1 + 0 e_2$

Matrix of linear transformation i.e T $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = A$

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0-\lambda & 1 \\ -1 & -\lambda \end{bmatrix} \quad \text{———} \quad (1)$$

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 = -1$$

$$\lambda = \pm i$$

Eigen Values are not real i.e $\pm i$

Q11 (i) For each of the following operators $T: R^3 \to R^3$, find all eigenvalues and a basis for each eigen space.

(i) $T(x, y, z) = (x+y+z, 2y+z, 2y+3z)$

Standard Basis for R^3
$e_1 = (1,0,0)$
$e_2 = (0,1,0)$
$e_3 = (0,0,1)$

$T(1,0,0) = (1+0+0, 2(0)+0, 2(0)+3(0))$

$= (1, 0, 0) = 1 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3$

$T(0,1,0) = (1, 2, 2) = 1 \cdot e_1 + 2e_2 + 2e_3$

$T(0,0,1) = (1, 1, 3) := \cdot e_1 + 1 \cdot e_2 + 3 \cdot e_3$

$\Rightarrow$ Matrix of T $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{pmatrix} = A$

$$A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 2 & 3-\lambda \end{bmatrix} \text{———} (1)$$

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$|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\{(2-\lambda)(3-\lambda)-2\} - 1(0-0) + 1(0-0) = 0$$

$$\Rightarrow (1-\lambda)(4-5\lambda+\lambda^2) = 0$$

$$\Rightarrow (1-\lambda)(\lambda-1)(\lambda-4) = 0$$

Eigen Values are $\lambda = 1, 1, 4$

Let Eigen Vector 'V' for $\lambda = 1$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$(A-\lambda I)V = 0 \Rightarrow$

Put $\lambda=1$ in ①
$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0x + y + z = 0$ $\Rightarrow y = -z$ let $z = b$ & $x = a$ (arbitrary values)

$0 \cdot x + 1 \cdot y + z = 0$

$0 \cdot x + 2y + 2z = 0$

$$\therefore V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ -b \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -b \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Eigen Vectors corresponding to $\lambda=1$ are $[1 \ 0 \ 0]^t$, $[0 \ -1 \ 1]^t$

These eigen vectors generates the sub space of R^3 called Eigen Space have basis $\{[1 \ 0 \ 0]^t, [0 \ -1 \ 1]^t\}$

2nd Method to find Eigen Vector

$(A-\lambda I)V = 0$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \underset{R}{\sim} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ Echelon form by $-R_1+R_2$, $-2R_1+R_3$

$y + z = 0$

Let $z = b$

& $x = a$

then $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ -b \\ b \end{bmatrix}$

Now Eigen Vector 'V' for $\lambda = 4$ $(A-\lambda I)V = 0$

Put $\lambda=4$ in ①
$$\begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x + y + z = 0$ $\Rightarrow 3x = y + z \Rightarrow \cancel{3}x = \cancel{3}y \Rightarrow x = y$

$-2y + z = 0$ $\Rightarrow z = 2y$

$2y - z = 0$ $\Rightarrow z = 2y$ Let

Let $x = a$ $\therefore V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ a \\ 2a \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Eigen Vector corresponding to $\lambda = 4$ is $[1 \ 1 \ 2]^t$

Eigen Vector generates a subspace of R^3 called Eigen Space of R^3 corresponding to $\lambda = 4$ having basis $\{[1 \ 1 \ 2]^t\}$

2nd Method

$$\begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \underset{R}{\sim} \begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ by R_2+R_3 Echelon form

$-3x + y + z = 0$

$-2y + z = 0$

$\Rightarrow z = 2y$

$\Rightarrow x = y$

Let $x = a$,

$$V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ a \\ 2a \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$