M.M. Chap. 8 ① M. Methods. 8

Exercise 8.1 - Chapter 08
Mathematical Methods
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# Infinite Series

**Sequence** is a fn $f(n)$ whose domain is set of natural numbers and whose range is a subset of real numbers.

e.g $f(n) = \frac{1}{n+1}$ Domain $= \{1, 2, 3, \ldots\}$ Range $= \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

Generally seq is denoted by $\{a_n\}$, and $a_n =$ $n^{th}$ term of seq.

**Infinite Sequence** has infinite number of terms, eg $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n} \ldots$

$\frac{1}{n} = a_n = n^{th}$ term

**Finite Sequence** has finite number of terms

**Convergence of a Sequence** An infinite seq $\{a_n\}$ is said to be convergent if $n^{th}$ term of seq tends to a definite real number '$l$' as '$n$' $\to \infty$

i.e $\boxed{\lim_{n\to\infty} a_n = l}$

**Divergence of a Sequence** A seq $\{a_n\}$ is said to be Divergent if $n^{th}$ term of seq tends to $\pm\infty$ as $n \to \infty$ i.e $\boxed{\lim_{n\to\infty} a_n = \pm\infty}$

**Bounded Above** If $B$ is a fixed number such that $a_n \leq B$ for every +ive integral value of '$n$', then we say $\{a_n\}$ seq is bounded above.

**Bounded Below** If $B$ is a fixed number such that $a_n \geq B$ for every +ive integral value of '$n$', then we say $\{a_n\}$ seq is bounded below.

**Bounded Sequence** is A seq which is bounded above & bounded below.

**Monotonic Sequences** A sequence $\{a_n\}$ is said to be

i) Non Decreasing if $a_{n+1} \geq a_n$ $\forall n$

ii) Non Increasing if $a_{n+1} \leq a_n$ $\forall n$

iii) strictly Increasing if $a_{n+1} > a_n$ $\forall n$

iv) strictly Decreasing if $a_{n+1} < a_n$ $\forall n$

**Note**

1) A convergent seq is Bounded but A Bounded seq need not be Convergent
2) A Bounded Monotonic Seq is Convergent
3) A Bounded above & Monotonic Increasing Seq is Convergent
4) A Bounded below & Monotonic Decreasing Seq is Convergent
5) An Unbounded Seq is Divergent

# EXERCISE 4.1 — N.M.

The nth term of a sequence is given. Determine whether the sequence converges or diverges. If it converges find its limit.

**Q(1)** $\dfrac{2}{\sqrt{n^2+3}}$

**SOL** We have $a_n = \dfrac{2}{\sqrt{n^2+3}}$

then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{2}{\sqrt{n^2+3}}$

$= \lim\limits_{n\to\infty} \dfrac{2}{n\sqrt{1+3/n^2}} = \dfrac{2}{\infty\sqrt{1+\frac{3}{\infty}}}$

$\boxed{\lim\limits_{n\to\infty} a_n = 0}$ ∵ '0' is definite number. So the sequence $\{a_n\}$ is Convergent, having limit '0'

**Q(2)** $\dfrac{(n-3)!}{(n-1)!}$

**SOL.** ∵ $a_n = \dfrac{(n-3)!}{(n-1)!}$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{(n-3)!}{(n-1)!}$

$= \lim\limits_{n\to\infty} \dfrac{(n-3)!}{(n-1)(n-2)(n-3)!}$

$= \lim\limits_{n\to\infty} \dfrac{1}{(n-1)(n-2)} = \dfrac{1}{\infty}$

$\boxed{\lim\limits_{n\to\infty} a_n = 0}$

⇒ The Sequence $\{a_n\}$ Converges to 0

**Q(3)** $1 + \dfrac{(-1)^n}{n}$

**SOL.** Here $a_n = 1 + \dfrac{(-1)^n}{n}$

then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(1 + \dfrac{(-1)^n}{n}\right)$

⇒ $\lim\limits_{n\to\infty} a_n = 1 + 0 = 1$

⇒ $\boxed{\lim\limits_{n\to\infty} a_n = 1}$ ∴ $\{a_n\}$ is Convergent Seq.

**Q.4** $a_n = \dfrac{\sqrt{n+1}}{n}$

**SOL.** ⇒ $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{\sqrt{n+1}}{n}$

$= \lim\limits_{n\to\infty} \dfrac{\sqrt{n}\sqrt{1+\frac{1}{n}}}{n}$

$= \lim\limits_{n\to\infty} \dfrac{\sqrt{1+\frac{1}{n}}}{\sqrt{n}} = 0$

$\boxed{\lim\limits_{n\to\infty} a_n = 0}$ ∵ 0 is definite number

So Sequence $\{a_n\}$ Converges to '0'.

**Q.5**

**SOL.** $a_n = n^{\frac{1}{n}}$

∴ $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n^{\frac{1}{n}}$ $\quad (\infty^0)$

So let $y = n^{\frac{1}{n}}$

⇒ $\ln y = \ln n^{\frac{1}{n}}$

$= \dfrac{1}{n}\ln n = \dfrac{\ln n}{n}$

⇒ $\lim\limits_{n\to\infty} \ln y = \lim\limits_{n\to\infty} \dfrac{\ln n}{n}$ $\quad \left(\dfrac{\infty}{\infty}\right)$

$= \lim\limits_{n\to\infty} \dfrac{1/n}{1}$ (L'Hospital Rule)

$= \lim\limits_{n\to\infty} \dfrac{1}{n} = 0.$

⇒ $\lim\limits_{n\to\infty} \ln y = 0$

ALTERNATE · 2

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^4+1}{4n^2-1}$$

$$= \lim_{n\to\infty} \frac{n^4\left(3+\frac{1}{n^4}\right)}{n^2\left(4-\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{n^2\left(3+\frac{1}{n^4}\right)}{4-\frac{1}{n^2}}$$

$$= \frac{\infty(3+0)}{4-0} = \infty$$

$\Rightarrow$ Then $\{a_n\}$ is a divergent sequence

Q.8 $a_n = \dfrac{\ln(n+1)}{\sqrt{n}}$

Sol

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln(n+1)}{\sqrt{n}} \quad \left(\frac{\infty}{\infty}\right)$$

(by L,Hospital)

$$= \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{2\sqrt{n}}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{n\to\infty} \frac{2\sqrt{n}}{n+1} \quad \left(\frac{\infty}{\infty}\right)$$

(by L,Hospital)

$$= \lim_{n\to\infty} \frac{\cancel{2}\cdot\frac{1}{\cancel{2}\sqrt{n}}}{1} = \lim_{n\to\infty} \frac{1}{\sqrt{n}}$$

$$\boxed{\lim_{n\to\infty} a_n = 0}$$

Sequence $\{a_n\}$ Converges to 0

Q.9 $a_n = \dfrac{e^n}{n}$

Sol.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{n} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{e^n}{1}$$

$$\lim_{n\to\infty} a_n = \infty$$

So Seq $\{a_n\}$ is Divergent

Q.10 Easy $a_n = \ln n - \ln(n+1)$

Sol

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\ln n - \ln(n+1)\right) \quad (\infty-\infty \text{ form})$$

$$= \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right)$$

$$= \ln \lim_{n\to\infty}\left(\frac{n}{n+1}\right) \quad \left(\frac{\infty}{\infty}\right)$$

$$= \ln \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right) = \ln 1 = 0$$

$$\boxed{\lim_{n\to\infty} a_n = 0}$$

$\therefore$ Seq $\{a_n\}$ converges to '0'

ALTERNATE Difficult

$\because a_n = \ln n - \ln(n+1)$ $(\infty - \infty$ form$)$

$$\Rightarrow a_n = \frac{\frac{1}{\ln(n+1)} - \frac{1}{\ln n}}{1/\ln n \cdot \ln(n+1)}$$

$\Rightarrow \lim_{n\to\infty} y = e^0 = 1$

or $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ Definite number

$$\boxed{\lim_{n\to\infty} a_n = 1}$$

$\Rightarrow$ The sequence $\{a_n\}$ converges to 1.

Easy **Q.6** $a_n = \dfrac{2^n}{(2n)!}$

**Sol**

$$\therefore = \frac{2^n}{(2n)(2n-1)(2n-2)(2n-3)(2n-4)\cdots 6\cdot5\cdot4\cdot3\cdot2\cdot1}$$

$$= \frac{2^n}{[(2n)(2n-2)(2n-4)(2n-6)\cdots 6\cdot4\cdot2]\,[(2n-1)(2n-3)\cdots 5\cdot3\cdot1]}$$

$$= \frac{\cancel{2^n}}{[\cancel{2^n}(n)(n-1)(n-2)(n-3)\cdots 3\cdot2\cdot1]\,[(2n-1)(2n-3)(2n-5)\cdots 5\cdot3\cdot1]}$$

$$a_n = \frac{1}{n!\,[(2n-1)(2n-3)(2n-5)\cdots 5\cdot3\cdot1]}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n!\,[(2n-1)(2n-3)\cdots 5\cdot3\cdot1]} = 0$$

$$\Rightarrow \boxed{\lim_{n\to\infty} a_n = 0}$$

$\therefore$ Seq $\{a_n\}$ converges to 0

Difficult 2nd Method Q6

**Sol** Let $a_n = \dfrac{2^n}{\lfloor 2n}$ we prove that given seq $\{a_n\}$ is bounded & decreasing

$$a_{n+1} = \frac{2^{n+1}}{\lfloor 2(n+1)} = \frac{2^{n+1}}{\lfloor 2n+2}$$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{\lfloor 2n+2}\cdot\frac{\lfloor 2n}{2^n} = \frac{\cancel{2^n}.2\;.\cancel{\lfloor 2n}}{(2n+2)(2n+1)\cancel{\lfloor 2n}\,\cancel{2^n}}$$

$$\frac{a_{n+1}}{a_n} = \frac{2}{(2n+2)(2n+1)} = \frac{1}{(n+1)(2n+1)} < 1 \;\forall n$$

$\Rightarrow a_{n+1} < a_n \therefore$ given seq $\{a_n\}$ is decreasing

$$\text{Now } a_n = \frac{2^n}{\lfloor 2n} = \frac{\cancel{2^n}}{\cancel{2^n}\lfloor n\,(2n-1)(2n-3)\cdots 3\cdot1}$$

$$= \frac{1}{\lfloor n\,(2n-1)(2n-3)\cdots 3\cdot1} \le 1$$

$\therefore$ Seq is bounded

$\because$ Seq is bounded & Decreasing Hence Convergent.

Now $\lim_{n\to\infty} a_n$

$$= \frac{1}{\lfloor n\,(2n-1)(2n-3)\cdots 3\cdot1}$$

$\Rightarrow 0$

So limit of seq is 0.

Difficult 2nd Method

**Q7** $a_n = \dfrac{3n^4+1}{4n^2-1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^4+1}{4n^2-1} \left(\frac{\infty}{\infty}\right)$$

using L' Hospital Rule

$$= \lim_{n\to\infty} \frac{12n^3}{8n} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{24n^2}{8} = \infty$$

which is not a definite number

So Sequence $\{a_n\}$ is divergent sequence.

**Q7** 2° ALTERNATE - Easy

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{3n^4+1}{4n^2-1}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\cancel{n^4}\left[3+\frac{1}{n^4}\right]}{\cancel{n^4}\left[\frac{4}{n^2}-\frac{1}{n^4}\right]}$$

$$= \frac{3+0}{0} = \infty$$

$$\Rightarrow \boxed{\lim_{n\to\infty} a_n = \infty}$$

$\{a_n\}$ is a divergent Seq.

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{1}{\ln(n+1)} - \frac{1}{\ln n}}{\frac{1}{(\ln n)(\ln(n+1))}} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{n\to\infty} \frac{(\cancel{n}+1) - \cancel{n}}{\frac{1}{\frac{1}{n}\ln(n+1) + \frac{1}{(n+1)}\ln n}} \quad (\text{by L'Hospital Rule})$$

$$= \lim_{n\to\infty} \frac{1}{\frac{1}{\frac{\ln(n+1)}{n} + \frac{\ln n}{n+1}}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\frac{(n+1)\ln(n+1) + n\ln n}{n(n+1)}}}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{n(n+1)}{(n+1)\ln(n+1) + n\ln n}} = \lim_{n\to\infty} \frac{(n+1)\ln(n+1) + n\ln n}{n^2+n}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot\frac{1}{n+1} + \ln(n+1) + n\cdot\frac{1}{n} + \ln n}{2n+1}$$

$$= \lim_{n\to\infty} \frac{1 + \ln(n+1) + 1 + \ln n}{2n+1} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n+1} + \frac{1}{n}}{2} = \lim_{n\to\infty} \frac{n + n+1}{2n(n+1)} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{1+1}{4n+2} = 0$$

$\Rightarrow$ the sequence $\{a_n\}$ converges to 0.

**Q.11** $a_n = \frac{\sin^2 n}{n}$

**Sol**

We know that

$$0 \leq \sin^2 n \leq 1$$

÷ by n $\Rightarrow \frac{0}{n} \leq \frac{\sin^2 n}{n} \leq \frac{1}{n}$

$$\Rightarrow \lim_{n\to\infty} \frac{0}{n} \leq \lim_{n\to\infty} \frac{\sin^2 n}{n} \leq \lim_{n\to\infty} \frac{1}{n}$$

$\frac{0}{\infty} = \frac{0}{\infty} = 0$ $\Rightarrow 0 \leq \lim_{n\to\infty} \frac{\sin^2 n}{n} \leq 0$

$$\Rightarrow \lim_{n\to\infty} \frac{\sin^2 n}{n} = 0$$

i.e $\boxed{\lim_{n\to\infty} a_n = 0}$ ∴ Seq $\{a_n\}$ converges to '0'

***ALTERNATE*** Easy

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin^2 n}{n}$$

∵ value of Sin lies between [-1, 1]

& value of $\sin^2 n$ lies between [0, 1]

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\text{any definite value in } [0,1]}{n}$$

$$= 0$$

$\Rightarrow \lim_{n\to\infty} a_n = 0$

$\Rightarrow$ Then sequence $\{a_n\}$ converges to '0'.

Q.12 $a_n = \frac{(2n)!}{(n!)^2}$

SOL.

then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(2n)!}{(n!)^2}$

or $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(2n)(2n-1)(2n-2)(2n-3)(2n-4)\cdots 6\cdot5\cdot4\cdot3\cdot2\cdot1}{(n!)(n!)}$

$= \lim_{n\to\infty} \frac{[(2n)(2n-2)(2n-4)\cdots 6\cdot4\cdot2][(2n-1)(2n-3)\cdots 5\cdot3\cdot1]}{(n!)(n!)}$

$= \lim_{n\to\infty} \frac{2^n[n(n-1)(n-2)\cdots 3\cdot2\cdot1][(2n-1)(2n-3)\cdots 5\cdot3\cdot1]}{(n!)(n!)}$

$= \lim_{n\to\infty} \frac{2^n \cancel{(n!)} (2n-1)(2n-3)(2n-5)\cdots 5\cdot3\cdot1}{\cancel{(n!)}[n(n-1)(n-2)\cdots 3\cdot2\cdot1]}$

$= \lim_{n\to\infty} \frac{2^n \cancel{n^n}\left[\left(2-\frac{1}{n}\right)\left(2-\frac{3}{n}\right)\left(2-\frac{5}{n}\right)\cdots\frac{3}{n}\cdot\frac{1}{n}\right]}{\cancel{n^n}\left[1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\cdots\frac{3}{n}\cdot\frac{2}{n}\cdot\frac{1}{n}\right]}$

$= \lim_{n\to\infty} \frac{2^n\left[\left(2-\frac{1}{n}\right)\left(2-\frac{3}{n}\right)\left(2-\frac{5}{n}\right)\cdots\cancel{\frac{1}{n}}\right]}{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\cdots\cancel{\frac{1}{n}}} = \infty$

$\boxed{\lim_{n\to\infty} a_n = \infty}$ $\Rightarrow$ $\{a_n\}$ seq Diverges

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Q. 13 $a_n = \left(\frac{2-n^2}{3+n^2}\right)^n$

SOL

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{2-n^2}{3+n^2}\right)^n = \lim_{n\to\infty} \frac{\cancel{(n^2)^n}}{\cancel{(n^2)^n}}\left[\frac{\frac{2}{n^2}-1}{\frac{3}{n^2}+1}\right]^n$

$= \lim_{n\to\infty}\left(\frac{0-1}{0+1}\right)^n = \lim_{n\to\infty}(-1)^n$

$\boxed{\lim_{n\to\infty} a_n = \pm 1}$ for odd & even '$n$'

$\lim_{n\to\infty} a_n$ does not ~~not~~ exist because limit is not unique

So Sequence is divergent.

**Q.14** $a_n = \frac{(\ln n)^2}{n}$

**Sol**, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(\ln n)^2}{n}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$= \lim_{n\to\infty} \frac{2\ln n \cdot \frac{1}{n}}{1}$ (using L'Hospital rule)

$= \lim_{n\to\infty} \frac{2\ln n}{n}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$= \lim_{n\to\infty} \frac{2\cdot\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{2}{n} = 0$

$\therefore \boxed{\lim_{n\to\infty} a_n = 0}$ $\therefore$ Seq converges to '0'.

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**Q15** $a_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$ Rationalize

**Sol**: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right) = \lim_{n\to\infty} \frac{\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\left(\sqrt{n+1}+\sqrt{n}\right)}$

$= \lim_{n\to\infty} \frac{\sqrt{n}\,(\cancel{n}+1-\cancel{n})}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$ $\left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{\cancel{\sqrt{n}}}{\left\{\sqrt{\frac{n+1}{n}}+1\right\}\cancel{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{n+1}{n}}+1}$

$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$

$\therefore \boxed{\lim_{n\to\infty} a_n = \frac{1}{2}}$ $\therefore$ Seq Converges to $\frac{1}{2}$.

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**Q16** $a_n = \frac{5^n + (-1)^n}{5^{n+1} + (-1)^{n+1}} = \frac{5^n + (-1)^n}{5\cdot 5^n + (-1)^n(-1)}$

**Sol**

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5^n + (-1)^n}{5\cdot 5^n + (-1)^n(-1)} = \lim_{n\to\infty} \frac{\cancel{5^n}\left(1 + \frac{(-1)^n}{5^n}\right)}{\cancel{5^n}\left(5 - \frac{(-1)^n}{5^n}\right)}$

$= \lim_{n\to\infty} \frac{1 + \frac{(-1)^n}{5^n}}{5 - \frac{(-1)^n}{5^n}} = \frac{1}{5}$

$\because (-1)^n = \begin{cases} -1, & \text{for } n \text{ is odd} \\ 1, & \text{for } n \text{ is even} \end{cases}$

$\boxed{\lim_{n\to\infty} a_n = \frac{1}{5}}$ $\therefore$ Seq $\{a_n\}$ converges to $\frac{1}{5}$

## Q.17 $a_n=(c^n+d^n)^{\frac{1}{n}}$, $d>c>0$

Difficult

**SOL**

then $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(d^n+c^n)^{\frac{1}{n}}$

$= \lim_{n\to\infty}(d^n)^{\frac{1}{n}}\left[1+\left(\frac{c}{d}\right)^n\right]^{\frac{1}{n}}$ $\qquad \frac{c}{d}<1$

$= \lim_{n\to\infty} d\left[1+\left(\frac{c}{d}\right)^n\right]^{\frac{1}{n}}$

$\therefore \lim_{n\to\infty} a_n = d(1+0)^0 = d$

$\boxed{\lim_{n\to\infty} a_n = d}$ $\therefore$ Seq $\{a_n\}$ is convergent

2nd Method

$a_n = (c^n+d^n)^{\frac{1}{n}}$

$= c\left(1+\left(\frac{d}{c}\right)^n\right)^{\frac{1}{n}}$ $\quad \infty^0$ form

Let $y = \left[1+\left(\frac{d}{c}\right)^n\right]^{\frac{1}{n}}$

$\ln y = \ln\left[1+\left(\frac{d}{c}\right)^n\right]^{\frac{1}{n}}$ ...

Solve yourself

Now $\lim_{n\to\infty}\left(\frac{c}{d}\right)^n = 0$, $\because \frac{c}{d}<1$

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## Q18 $a_n = \frac{5^n}{(n+1)^2}$

**Sol** $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{5^n}{(n+1)^2}$ $\left(\frac{\infty}{\infty}\right)$

L-Hospital Rule. $\frac{d}{dx}(a^x) = a^x \ln a$

$= \lim_{n\to\infty}\frac{5^n(\ln 5)}{2(n+1)}$ $\left(\frac{\infty}{\infty}\right)$

L-Hospital Rule

$= \lim_{n\to\infty}\frac{(5^n \ln 5).(\ln 5)}{2}$

$(\ln 5)$ is const.

$= \frac{(\ln 5)^2}{2}\lim_{n\to\infty} 5^n$

$\boxed{\lim_{n\to\infty} a_n = \infty}$

$\therefore \{a_n\}$ is Divergent

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Easy

## Q 19 $a_n = 1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\cdots\cdots+\frac{1}{\underline{|n}}$

We know $e^x = 1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\cdots\cdots\frac{x^n}{\underline{|n}}$ as $n\to\infty$ (expansion up to n terms)

Put $x=1$,

$e = 1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\cdots\cdots\frac{1}{\underline{|n}}$ as $n\to\infty$ (up to n terms)

$e = 1+a_n$

$e-1 = a_n$

$a_n = \lim_{n\to\infty}(e-1) = \boxed{e-1}$

Seq $\{a_n\}$ converges to $e-1$

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Q19 $a_n = 1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\cdots\frac{1}{\underline{|n}}$

$a_{n+1} = 1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\cdots\cdots\frac{1}{\underline{|n}}+\frac{1}{\underline{|n+1}}$

$a_{n+1}-a_n = \frac{1}{\underline{|n+1}}$

$a_{n+1}-a_n > 0$

$a_{n+1} > a_n$ $\therefore$ given Seq $\{a_n\}$ is Increasing

Now we prove $\{a_n\}$ is Bounded.

$a_n = 1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\cdots\cdots+\frac{1}{\underline{|n}}$

$= 1+\frac{1}{1\cdot 2}+\frac{1}{3\cdot 2\cdot 1}+\cdots\cdots+\frac{1}{n\cdot(n-1)\cdots\cdots 3\cdot 2\cdot 1}$

$< 1+\frac{1}{2}+\frac{1}{2\cdot 2}+\cdots\cdots+\frac{1}{2\cdot 2\cdot 2\cdots\cdots 2\cdot 2\cdot 1}$

$= 1+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots\frac{1}{2^{n-1}}$

$= \frac{1\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$ $\qquad$ G. Series $a=1$, $r=\frac{1}{2}$, $\frac{a(1-r^n)}{1-r}$

$= 2\left(1-\frac{1}{2^n}\right)$

$= 2-\frac{1}{2^{n-1}}$

$a_n < 2$ Seq $\{a_n\}$ is bounded above by 2 and increasing so converges

Now $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\cdots\cdots\frac{1}{\underline{|n}}\right)$

$= \lim_{n\to\infty}\left(\left(1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\cdots\frac{1}{\underline{|n}}\right)-1\right)$

$\boxed{\lim_{n\to\infty} a_n = e-1}$

So the Seq $\{a_n\}$ has limit $e-1$ and it is convergent

**Q.20** $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2+a_n}$, $n \geq 1$

**Sol** Since $a_{n+1} = \sqrt{2+a_n}$, Since $a_1 = \sqrt{2}$
So $a_1 < 2$

for $n=1$: So $a_2 = \sqrt{2+a_1}$

$\because a_1 < 2$: $a_2 < \sqrt{2+2}$

$a_2 < \sqrt{4} \Rightarrow a_2 < 2$

Now for $n=2$: $a_3 = \sqrt{2+a_2}$

$\because a_2 < 2$: So $a_3 < \sqrt{2+2}$

$a_3 < \sqrt{4} \Rightarrow a_3 < 2$

and so on, so $a_n < 2$

Similarly $a_{n+1} < 2$

$\Rightarrow$ Sequence is bounded above by 2.

Now since $a_{n+1} = \sqrt{2+a_n}$

$\therefore (a_{n+1})^2 = 2 + a_n$ substracting $(a_n)^2$

$\Rightarrow (a_{n+1})^2 - (a_n)^2 = 2 + a_n - a_n^2$ $+a_n - a_n$

$= 2 + 2a_n - a_n - a_n^2$

$= 2(1+a_n) - a_n(1+a_n)$

$= (2-a_n)(1+a_n)$

$> 0$ $\quad \because a_n < 2 \therefore 2 - a_n > 0$

$\Rightarrow (a_{n+1})^2 - (a_n)^2 > 0$

$\Rightarrow (a_{n+1})^2 > (a_n)^2 \Rightarrow a_{n+1} > a_n$

$\Rightarrow$ The sequence is monotonically increasing

Since the sequence is bounded above and increasing, So Sequence is Convergent.

Since the given sequence is convergent. So its limit will exist and let its limit is $a > 0$

i.e. $\lim\limits_{n\to\infty} a_{n+1} = a$

$\lim\limits_{n\to\infty} \sqrt{2+a_n} = a$

If $\lim\limits_{n\to\infty} a_{n+1} = a$ then $\lim\limits_{n\to\infty} a_n = a$ also

$\Rightarrow \sqrt{2+a} = a$

squaring $\Rightarrow 2 + a = a^2$

$\Rightarrow a^2 - a - 2 = 0 \Rightarrow a = \dfrac{+1 \pm \sqrt{1-4\cdot1\cdot(-2)}}{2} = \dfrac{1\pm\sqrt{9}}{2} = \dfrac{1\pm3}{2}$

$\Rightarrow a = -1, 2$

$\because a > 0$, $\therefore a = 2$, is required limit of Sequence.

**Note** A monotonic increasing seq which is bounded above is Convergent.

ii) If $a_{n+1} \geq a_n$ then $\{a_n\}$ is Mon. Inc Seq.

iii) If $a_n < B$ where B is fixed number then $\{a_n\}$ is bounded above.

(20) Easy 2nd Method: $a_{n+1} = \sqrt{2+a_n}$

Let $\lim\limits_{n\to\infty} a_n = l$ — (I)

then $\lim\limits_{n\to\infty} a_{n+1} = l$ — (II)

$\Rightarrow \lim\limits_{n\to\infty} \sqrt{2+a_n} = l$ $\because$ (given)

squaring $\lim\limits_{n\to\infty}(2+a_n) = l^2$

$2 + l = l^2$

$l^2 - l - 2 = 0$ Q Eq

$l^2 - 2l + l - 2 = 0$

$l(l-2) + 1(l-2) = 0$

$(l-2)(l+1) = 0$

$l = 2, -1$ $\quad \therefore \boxed{l = 2}$

since the seq is a +ive terms seq so $l = -1$ is not possible

$\therefore \lim\limits_{n\to\infty} a_n = l$

$\lim\limits_{n\to\infty} a_n = 2.$