EXERCISE 8.3

Apply "RATIO TEST" to determine whether the given series converges or diverges (Problems 1—10)

Q.1 $\sum_{0}^{\infty} \frac{2^n}{(2n)!}$

Sol. $\Rightarrow a_n = \frac{2^n}{(2n)!}$

$\Rightarrow a_{n+1} = \frac{2^{n+1}}{(2n+2)!}$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{2^n}$$

$$= \lim_{n\to\infty} \frac{2\cdot \cancel{2^n}}{(2n+2)(2n+1)\cancel{2n!}} \cdot \frac{\cancel{(2n)!}}{\cancel{2^n}}$$

$$= \lim_{n\to\infty} \frac{2}{(2n+2)(2n+1)} = \boxed{0<1}$$

$\therefore$ Hence Convergent Series

RATIO TEST

Let $\sum_{1}^{\infty} a_n$ be a +ive term series and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, where l is non-negative real no. or ∞

(i) if $l<1$, series $\sum_{1}^{\infty} a_n$ converges

(ii) if $l>1$ or ∞, series $\sum_{1}^{\infty} a_n$ diverges

(iii) if $l=1$, test fails

Note: If Root test, Ratio test and Integral test do not hold then use B.C.T

Q.2 $\sum_{1}^{\infty} \frac{n!}{n^n}$

Sol. $a_n = \frac{n!}{n^n} \Rightarrow a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{(n+1)\cancel{n!}\, n^n}{(n+1)^{n+1}\cdot \cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}\, n^n}{\cancel{(n+1)}(n+1)^n} = \lim_{n\to\infty} \frac{1}{(n+1)^n/n^n} = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \boxed{\frac{1}{e}<1}$$

$\therefore$ Hence Convergent Series ($\because e = 2.71+$)

Q.3 $\sum_{1}^{\infty} 7^n / n(5^{n+1})$

Sol. $a_n = \frac{7^n}{n(5^{n+1})}$

$a_n = \frac{7^n}{5n(5^n)}$

$a_{n+1} = \frac{7^{n+1}}{5(n+1)5^{n+1}} = \frac{7\cdot 7^n}{25(n+1)5^n}$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{7\cdot \cancel{7^n}}{\cancel{25}\,5(n+1)\cancel{5^n}} \cdot \frac{\cancel{5}\,n(\cancel{5^n})}{\cancel{7^n}} = \lim_{n\to\infty} \frac{7n}{5(n+1)}$$

$$= \lim_{n\to\infty} \frac{7}{5(1+1/n)} = \boxed{\frac{7}{5}>1}$$

$\therefore \sum a_n$ diverges

Q.4 $\sum_{1}^{\infty} \frac{n}{n^2+1}$

$a_n = \lim_{n\to\infty} \frac{n}{n^2+1}$
$= \lim_{n\to\infty} \frac{n}{n^2(1+\frac{1}{n^2})}$
$= \lim_{n\to\infty} \frac{1}{n(1+\frac{1}{n^2})}$
$= \frac{1}{\infty}$
$a_n = 0$ so apply some other test.

SoL. $a_n = \frac{n}{n^2+1} \Rightarrow a_{n+1} = \frac{n+1}{(n+1)^2+1} = \frac{n+1}{n^2+2n+2}$

Then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n^2+2n+2} \times \frac{n^2+1}{n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n^2}\left[1+\frac{2}{n}+\frac{2}{n^2}\right]} \times \frac{\cancel{n^2}(1+\frac{1}{n^2})}{\cancel{n}}$$

$$= \lim_{n\to\infty} \frac{(1+\frac{1}{n})(1+\frac{1}{n^2})}{(1+\frac{2}{n}+\frac{2}{n^2})} = 1$$ Test fails to determine the convergence or divergence

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$

Note we can also solve it by integral test. $\because \int_1^{\infty} \frac{x}{x^2+1}\,dx$ can be integrated easily.

So we use L.C.T.
Let $\sum b_n = \sum \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left[\frac{n}{n^2+1} \times \frac{n}{1}\right] = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}}\left[\frac{1}{1+\frac{1}{n^2}}\right] = 1 \neq 0$$

$\Rightarrow$ both series behave alike. Since $\sum b_n = \sum \frac{1}{n}$ is divergent, So $\sum a_n$ is also divergent.

Q.5 $\sum_{1}^{\infty} \frac{(2n)!}{4^n}$

SoL $\because a_n = \frac{(2n)!}{4^n} \Rightarrow a_{n+1} = \frac{(2n+2)!}{4^{n+1}} = \frac{(2n+2)!}{4^n \cdot 4}$ $\quad \because 4^{n+1} = 4^n 4$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+2)!}{4 \cdot 4^n} \times \frac{4^n}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cancel{2n!}}{4 \cdot \cancel{4^n}} \times \frac{\cancel{4^n}}{\cancel{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{4} = \infty$$

Hence Divergent Series

Q.6 $\sum_{1}^{\infty} \frac{2^n}{n(n+2)}$

Sol. $\Rightarrow a_n = \frac{2^n}{n(n+2)}$, and $a_{n+1} = \frac{2^{n+1}}{(n+1)(n+3)}$

$= \frac{2 \cdot 2^n}{(n+1)(n+3)}$

$$\underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = \underset{n\to\infty}{Lt} \frac{2 \cdot \cancel{2^n}}{(n+1)(n+3)} \cdot \frac{n(n+2)}{\cancel{2^n}}$$

$$= \underset{n\to\infty}{Lt} \frac{2\,\cancel{n^2}\left(1+\frac{2}{n}\right)}{\cancel{n^2}\left[1+\frac{1}{n}\right]\left[1+\frac{3}{n}\right]} = \underset{n\to\infty}{Lt} \frac{2\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{n}\right)}$$

$= \boxed{2 > 1}$ Hence divergent series

Q.7 $\sum_{1}^{\infty} \frac{(n+1)(n+2)}{n!}$

Sol. $\Rightarrow a_n = \frac{(n+1)(n+2)}{n!}$

$\Rightarrow a_{n+1} = \frac{(n+2)(n+3)}{(n+1)!}$

$$\underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = \underset{n\to\infty}{Lt} \frac{(\cancel{n+2})(n+3)}{(n+1)!} \times \frac{n!}{(n+1)(\cancel{n+2})}$$

$$= \underset{n\to\infty}{Lt} \frac{(n+3)\,\cancel{n!}}{(n+1)(n+1)\,\cancel{n!}} = \underset{n\to\infty}{Lt} \frac{\left(1+\frac{3}{n}\right)}{n\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}$$

$= \boxed{0 < 1}$ Hence Convergent Series

Q.8 $n^3 e^{-n^4}$

Sol. $a_n = n^3 e^{-n^4}$

$a_{n+1} = (n+1)^3 e^{-(n+1)^4}$

$$\underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^3 e^{-(n+1)^4}}{n^3 e^{-n^4}} = \lim_{n\to\infty} \frac{\cancel{n^3}\left(1+\frac{1}{n}\right)^3 e^{-(n+1)^4}}{\cancel{n^3}\; e^{-n^4}}$$

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$$= \lim_{n\to\infty}\left(1+\frac{1}{n^3}\right)^3 \cdot \lim_{n\to\infty} \frac{e^{-(n+1)^4}}{e^{-n^4}}$$

$$= 1 \cdot \lim_{n\to\infty} e^{-(\cancel{n^4}+4n^3+6n^2+1-\cancel{n^4}+4n)}$$

$$= \lim_{n\to\infty} \frac{1}{e^{4n^3+6n^2+1+4n}} = \boxed{0<1}$$

Hence Convergent Series

$$(1+n)^4 = 1+4n+\frac{4\cdot 3}{2\cdot 1}n^2 + \frac{4\cdot 3\cdot 2}{3\cdot 2\cdot 1}n^3 + n^4 = 1+4n+6n^2+4n^3+n^4$$

Q.9 $\sum_{1}^{\infty} \frac{(n!)^2}{(3n)!}$

SoL. $\Rightarrow a_n = \frac{(n!)^2}{(3n)!}, \quad a_{n+1} = \frac{[(n+1)!]^2}{(3n+3)!}$

using ratio test. i.e

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{[(n+1)!]^2}{(3n+3)!} \times \frac{(3n)!}{(n!)^2}$$

$$= \lim_{n\to\infty} \frac{[(n+1)\,\cancel{n!}]^2}{(\cancel{n!})^2} \times \frac{\cancel{(3n)!}}{(3n+3)(3n+2)(3n+1)\,\cancel{(3n)!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(1+\frac{1}{n}\right)^2}{n^{\cancel{3}}\left(3+\frac{3}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2}{n\left(3+\frac{3}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{1}{n}\right)} = \frac{1}{\infty} = \boxed{0<1}$$

Hence Convergent Series

Q.10 $\sum_{1}^{\infty} \frac{(n+2)!}{4!\, n!\, 2^n}$

SoL Here $a_n = \frac{(n+2)!}{4!\, n!\, 2^n}, \quad a_{n+1} = \frac{(n+3)!}{4!\,(n+1)!\,2^{n+1}}$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+3)!\quad 4!\, n!\, 2^n}{4!\,(n+1)!\,2^{n+1}\,(n+2)!}$$

$$= \lim_{n\to\infty} \frac{(n+3)\,\cancel{(n+2)!}\,\cancel{4!}\,n!\,\cancel{2^n}}{\cancel{4!}\,(n+1)!\,2\cdot\cancel{2^n}\,\cancel{(n+2)!}} = \lim_{n\to\infty} \frac{(n+3)\,\cancel{n!}}{2\cdot(n+1)\,\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{3}{n}\right)}{2\cancel{n}\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{1+\frac{3}{n}}{2\left(1+\frac{1}{n}\right)} = \boxed{\frac{1}{2}<1}$$

Hence Convergent Series

(15) Easy

Ratio Test 2001

$a_n = \frac{3^n}{n^3}$

$a_{n+1} = \frac{3^{n+1}}{(n+1)^3}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{(n+1)^3} \cdot \frac{n^3}{3^n}$

$= \lim_{n\to\infty} \frac{3\cancel{n^3}}{\cancel{n^3}(1+\frac{1}{n})^3}$

$= 3 > 1$

Dgt

Q.15 $\sum_{1}^{\infty} \frac{3^n}{n^3}$

Sol. Here $a_n = \frac{3^n}{n^3} \Rightarrow (a_n)^{\frac{1}{n}} = \left(\frac{3^n}{n^3}\right)^{\frac{1}{n}} = \frac{3}{n^{\frac{3}{n}}}$

$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{3}{(n^{\frac{1}{n}})^3}$

$= \frac{3}{(\lim_{n\to\infty} n^{\frac{1}{n}})^3} = \boxed{\frac{3}{1} > 1}$ Hence Divergent

Proof $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ (∞^0)

Let $y = n^{\frac{1}{n}}$

$\ln y = \frac{1}{n} \ln n$

$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln n}{n}$ $\left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{1/n}{1}$ L'Hospital

$\lim_{n\to\infty} \ln y = 0$

$\ln(\lim_{n\to\infty} y) = 0$

$\lim_{n\to\infty} y = e^0$ Antilog

$\boxed{\lim_{n\to\infty} n^{\frac{1}{n}} = 1}$

Q.16 $\sum_{1}^{\infty} n\left(\frac{\pi}{n}\right)^n$

$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \left[n\left(\frac{\pi}{n}\right)^n\right]^{\frac{1}{n}}$

$= \lim_{n\to\infty} n^{\frac{1}{n}}\left(\frac{\pi}{n}\right)$

$= \lim_{n\to\infty} n^{\frac{1}{n}} \cdot \lim_{n\to\infty} \frac{\pi}{n}$

$= 1 \cdot 0$

$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = 0 < 1 \therefore \sum a_n$ Converges

In Problems 17–36, apply any approximate test to determine the convergence or divergence of the Series:

Root.T **Q.17** $\sum_{2}^{\infty} \frac{e^n}{(\ln n)^n}$

Sol. Here $a_n = \frac{e^n}{(\ln n)^n} \Rightarrow (a_n)^{\frac{1}{n}} = \left[\frac{e^n}{(\ln n)^n}\right]^{\frac{1}{n}} = \frac{e}{\ln n}$

$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{e}{\ln n} = \frac{e}{\infty} = 0 < 1 \Rightarrow \sum a_n$ is convergent

2002A **Q.18** $\sum_{1}^{\infty} \frac{1^n + 2^n}{3^n}$

Sol $\Rightarrow a_n = \frac{1^n + 2^n}{3^n}$ $a_{n+1} = \left(\frac{1^{n+1} + 2^{n+1}}{3^{n+1}}\right)$

$\because 1^n = 1^{n+1} = 1$

$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1^{n+1} + 2^{n+1}}{3^{n+1}} \times \frac{3^n}{1^n + 2^n} = \lim_{n\to\infty} \frac{1 + 2^{n+1}}{3(1+2^n)}$

Apply Cauchy's Root test to determine whether the given series converges or diverges (Problems 11—16)

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Q.11 $\sum_{1}^{\infty}\left(\frac{3n+2}{2n-1}\right)^n$

SOL. $\Rightarrow a_n = \left(\frac{3n+2}{2n-1}\right)^n$

Then using Cauchy's Root-test i.e

$$\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left(\frac{3n+2}{2n-1}\right)^{\cancel{n}}\right]^{\frac{1}{\cancel{n}}}$$

$$= \lim_{n\to\infty}\frac{3n+2}{2n-1} = \lim_{n\to\infty}\frac{\cancel{n}(3+\frac{2}{n})}{\cancel{n}(2-\frac{1}{n})}$$

$$= \lim_{n\to\infty}\frac{3+\frac{2}{n}}{2-\frac{1}{n}} = \boxed{\frac{3}{2} > 1} \Rightarrow \Sigma a_n \text{ diverges.}$$

ROOT TEST

let $\sum_{1}^{\infty} a_n$ is +ve terms series and let $\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = l$

l is nonnegative real no. or ∞

(i) if $l<1$, Σa_n converges

(ii) if $l>1$ or ∞, Σa_n diverges

(iii) if $l=1$ . . . Test fails

Q.12 $\sum_{1}^{\infty}\frac{1}{n^n}$

SOL. Here $a_n = \frac{1}{n^n}$

$$\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1}{n^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}(n^{-n})^{\frac{1}{n}} = \lim_{n\to\infty} n^{-1}$$

$$= \lim_{n\to\infty}\frac{1}{n} = \boxed{0<1} \Rightarrow \Sigma a_n \text{ Converges.}$$

Q.13 $\sum_{1}^{\infty}\left(\frac{n}{10}\right)^n$

SOL Here $a_n = \left(\frac{n}{10}\right)^n \Rightarrow (a_n)^{\frac{1}{n}} = \left[\left(\frac{n}{10}\right)^n\right]^{\frac{1}{n}} = \frac{n}{10}$

So by R. Test $\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{10} = \boxed{\infty > 1}$ $\therefore \Sigma a_n$ Diverges

Q.14 $\sum_{1}^{\infty}\left(\frac{n}{1+n^3}\right)^n$

SOL. Here $a_n = \left(\frac{n}{1+n^3}\right)^n \Rightarrow (a_n)^{\frac{1}{n}} = \left[\left(\frac{n}{n^3+1}\right)^n\right]^{\frac{1}{n}} = \frac{n}{n^3+1}$

$$\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n^3+1} = \lim_{n\to\infty}\frac{1}{n^2\left(1+\frac{1}{n^3}\right)} = \boxed{0<1} \Rightarrow \Sigma a_n \text{ Converges.}$$

$$= \lim_{n\to\infty} \frac{\cancel{2^n}\left(\frac{1}{2^n}+2\right)}{3\cdot\cancel{2^n}\left(\frac{1}{2^n}+1\right)}$$

$$= \frac{0+2}{3(0+1)} = \boxed{\frac{2}{3}<1}$$

Hence Convergent

2nd Method Q18

$$a_n = \frac{1^n+2^n}{3^n}$$

$$a_n = \frac{1^n}{3^n}+\frac{2^n}{3^n}$$

$$a_n = b_n + c_n$$

$$\Sigma a_n = \Sigma b_n + \Sigma c_n$$

$\Sigma b_n = \Sigma\left(\frac{1}{3}\right)^n$ is cgt $\because$ IGS $r=\frac{1}{3}<1$

$\Sigma c_n = \Sigma\left(\frac{2}{3}\right)^n$ is cgt $\because$ IGS $r=\frac{2}{3}<1$

$\therefore \Sigma a_n$ is convergent being sum of two cgt series.

Root

Q.19 $\sum_1^\infty \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$

SOL

$$\Rightarrow a_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$$

$$\Rightarrow \lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\left[\left(1+\frac{1}{\sqrt{n}}\right)^{n^{\frac{3}{2}}}\right]^{\frac{1}{n}}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\frac{n^{3/2}}{n}}}$$

Thus $\lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \boxed{\frac{1}{e}<1}$

Q20 $\lim_{n\to\infty} a_n$

$$= \lim_{n\to\infty} \frac{\ln n}{e^n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{e^n}$$

$$= \lim_{n\to\infty} \frac{1}{ne^n}$$

$= 0$. Does not know cgs or dgs. So use some other Test

Ratio

Q.20 $\sum_1^\infty \frac{\ln n}{e^n}$

Sol

$$a_n = \frac{\ln(n)}{e^n}$$

$$a_{n+1} = \frac{\ln(n+1)}{e^{n+1}}$$

then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\ln(n+1)}{e^n\cdot e} \times \frac{\cancel{e^n}}{\ln n}$

(L'Hospital Rule) $= \frac{1}{e}\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{1}{e}\lim_{n\to\infty}\frac{n}{n+1}$

$$= \frac{1}{e}\lim_{n\to\infty}\left(\frac{\cancel{n}}{\cancel{n}\left(1+\frac{1}{n}\right)}\right) = \frac{1}{e(1+0)}$$

$= \boxed{\frac{1}{e}<1}$ Hence Convergent

Root

2nd Method Q20

$\sum_1^\infty \frac{\ln n}{e^n}$, $\because \ln n < n$

$$\therefore \Rightarrow \frac{\ln n}{e^n} < \frac{n}{e^n}$$

Let $a_n = \frac{n}{e^n}$

$$\Rightarrow \lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \frac{\lim_{n\to\infty} n^{\frac{1}{n}}}{(e^n)^{\frac{1}{n}}}$$

$\left(\because \lim_{n\to\infty} n^{\frac{1}{n}} = 1\right)$

$$= \frac{1}{e} < 1$$

$\Rightarrow \Sigma\frac{n}{e^n}$ is cgt

$\Rightarrow \Sigma\frac{\ln n}{e^n}$ is also cgt.

B. Compare Test

cgt $\because$ cgt

Ratio Test

Q.21 $\sum_{1}^{\infty} \frac{(n!)^2 \, 2^n}{(2n+2)!}$

Sol

$\Rightarrow a_n = (n!)^2 \, 2^n / (2n+2)!$

$\Rightarrow a_{n+1} = \frac{[(n+1)!]^2 \, (2^{n+1})}{(2n+4)!}$

$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{[(n+1)!]^2 (2 \cdot 2^n)}{(2n+4)!} \times \frac{(2n+2)!}{(n!)^2 2^n}$

$= \lim_{n\to\infty} \frac{(n+1)^2 \cancel{(n!)^2} (2 \cdot \cancel{2^n}) \times \cancel{(2n+2)!}}{(2n+4)(2n+3)\cancel{(2n+2)!}\cancel{(n!)^2}\cancel{2^n}}$

$= \lim_{n\to\infty} \frac{2(n+1)^2}{(2n+4)(2n+3)} = \lim_{n\to\infty} \frac{2\left(1+\frac{1}{n}\right)^2}{\left(2+\frac{4}{n}\right)\left(2+\frac{3}{n}\right)}$

$= \frac{2}{4} = \boxed{\frac{1}{2} < 1}$ Hence Convergent series

Q22 Can be solved Also by Integral Test

LCT

Q.22 $\sum_{1}^{\infty} \frac{\tan^{-1} n}{n^2}$

Sol Here $a_n = \frac{\tan^{-1} n}{n^2}$

Let $b_n = \frac{1}{n^2}$ $\quad 0-2 = \frac{1}{n^2}$

using L. Comparison test

i.e $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\tan^{-1} n}{n^2}}{1/n^2}$

$= \lim_{n\to\infty} \tan^{-1} n$

$= \tan^{-1} \infty = \frac{\pi}{2} = l \neq 0$

($\frac{\pi}{2}$ is non ~~zero~~ finite)

by $\Sigma b_n = \Sigma \frac{1}{n^2}$

is convergent by Euler Series

so Σa_n is convergent

Root **Q. 23** $\sum_{1}^{\infty}\left(\frac{5n}{2n+1}\right)^{3n}$,

SoL. Here $a_n = \left(\frac{5n}{2n+1}\right)^{3n}$,

apply root test i.e

$$\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\left(\frac{5n}{2n+1}\right)^{3n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{5n}{2n+1}\right)^{3}$$

$$= \lim_{n\to\infty}\left(\frac{5n}{n(2+\frac{1}{n})}\right)^{3}$$

$$= \left(\frac{5}{2}\right)^{3} > 1$$ Hence divergent

Root **Q. 24** $\sum_{1}^{\infty}\left(\frac{n!}{n^n}\right)^{n}$

SoL. Here $a_n = \left(\frac{n!}{n^n}\right)^{n}$

then by root test

$$\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\left(\frac{n!}{n^n}\right)^{n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{n\cdot n\cdot n\cdots n\cdot n}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{2}{n}\right)\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\cdots\cdots\left(\frac{3}{n}\right)\left(\frac{2}{n}\right)\left(\frac{1}{n}\right)$$

$$= 1\cdot 1\cdot 1\cdots\ 0\cdot 0\cdot 0 = \boxed{0<1}$$ Hence Convergent Series

LCT **Q. 25** $\sum_{1}^{\infty}\frac{5\sqrt{n}+1}{\sqrt{n^3-2n^2+3}}$

SoL. Here $a_n = \frac{5\sqrt{n}+1}{\sqrt{n^3-2n^2+3}}$

$\frac{1}{2}-\frac{3}{2}=-1$

$b_n = n^{-1} = \frac{1}{n}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{5\sqrt{n}+1}{\sqrt{n^3-2n^2+3}}\cdot\frac{n}{1}$$

$$= \lim_{n\to\infty}\frac{n^{3/2}\left(5+\frac{1}{\sqrt{n}}\right)}{n^{3/2}\left(1-\frac{2}{n}+\frac{3}{n^3}\right)^{\frac{1}{2}}} = \frac{5}{1} \neq 0$$

By LCT

Σa_n and Σb_n behaves alike.

$\because \Sigma b_n = \Sigma\frac{1}{n}$ dgs so Σa_n dgs. (Harmonic Series is dgt.)

$$= \lim_{n\to\infty} \frac{\sin\frac{1}{n^{1/2}}}{\left(1-\frac{2}{n}+\frac{3}{n^3}\right)^{1/2}} = 5 \neq 0$$

but Σb_n diverges. Hence by L. Comparison test (∵ Harmonic Series is dgt)

Q.26 $\sum_0^\infty \frac{2^n+n}{(n+1)!}$

Sol. $\Rightarrow a_n = \frac{2^n+n}{(n+1)!} \Rightarrow a_{n+1} = \frac{2^{n+1}+n+1}{(n+2)!}$

or $a_{n+1} = \frac{2\cdot 2^n+n+1}{(n+2)(n+1)!}$

By Ratio Test

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(2\cdot 2^n+n+1)\cancel{(n+1)!}}{(n+2)\cancel{(n+1)!}\,(2^n+n)}$$

$+n-n$

$$= \lim_{n\to\infty}\frac{2\cdot 2^n+2n+1-n}{(n+2)(2^n+n)}$$

$$= \lim_{n\to\infty}\left(\frac{2\cancel{(2^n+n)}}{(n+2)\cancel{(2^n+n)}} + \frac{1-n}{(n+2)(2^n+n)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2}{n(1+\frac{2}{n})} + \frac{\cancel{n}(-1+\frac{1}{n})}{\cancel{n}(1+\frac{2}{n})(2^n+n)}\right)$$

$$= 0 - \frac{1}{\infty}$$

$$= \boxed{0<1}$$

Hence Σa_n Converges

ALTERNATE **Note:** Here we use the following Theorem i.e.

If $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$ are convergent series with sums S and T

Then $\sum_1^\infty (a_n+b_n)$ and $\sum_1^\infty (a_n-b_n)$ are convergent and sums of the series are

$$\sum_1^\infty (a_n+b_n) = \sum_1^\infty a_n + \sum_1^\infty b_n = S+T$$

and $$\sum_1^\infty (a_n-b_n) = \sum_1^\infty a_n - \sum_1^\infty b_n = S-T$$

(ii) If $\sum_1^\infty a_n$ Converges and $\sum_1^\infty b_n$ diverges then $\sum_1^\infty (a_n+b_n)$ diverges"

Imp Q26 $\displaystyle\sum_0^\infty \frac{2^n+n}{(n+1)!}$

$a_n = \dfrac{2^n+n}{(n+1)!} \Rightarrow a_n = \dfrac{2^n}{(n+1)!} + \dfrac{n}{(n+1)!} \Rightarrow a_n = b_n + c_n$

where $b_n = \dfrac{2^n}{(n+1)!}$ & $c_n = \dfrac{n}{(n+1)!}$

$\Rightarrow b_{n+1} = \dfrac{2\cdot 2^n}{(n+2)!}$ $\quad c_{n+1} = \dfrac{n+1}{(n+2)!}$

$b_{n+1} = \dfrac{2\cdot 2^n}{(n+2)(n+1)!}$ $\quad c_{n+1} = \dfrac{(n+1)}{(n+2)(n+1)!}$

Then by ratio test

$\lim\limits_{n\to\infty} \dfrac{b_{n+1}}{b_n} = \lim\limits_{n\to\infty} \dfrac{2\cdot \cancel{2^n}}{(n+2)\cancel{(n+1)!}} \times \dfrac{\cancel{(n+1)!}}{\cancel{2^n}} = \lim\limits_{n\to\infty} \dfrac{2}{n+2} = \boxed{0<1}$ $\therefore \sum b_n$ is Cgt

and $\lim\limits_{n\to\infty} \dfrac{c_{n+1}}{c_n} = \lim\limits_{n\to\infty} \dfrac{\cancel{n}}{(n+2)\cancel{(n+1)!}} \times \dfrac{\cancel{(n+1)!}}{n}$

$= \lim\limits_{n\to\infty} \dfrac{1}{n+2} = \boxed{0<1} \Rightarrow \therefore \sum c_n$ is convergent

$\because a_n = b_n + c_n \Rightarrow \sum_1^\infty a_n = \sum_1^\infty b_n + \sum_1^\infty c_n$

Since both $\sum_1^\infty b_n$ and $\sum_1^\infty c_n$ are convergent

So $\sum_1^\infty b_n + \sum_1^\infty c_n$ is also convergent

$\Rightarrow \sum_1^\infty a_n$ is convergent

Q.27 $\sum_1^\infty \frac{\sqrt{n}}{2^n} = \sum_1^\infty \frac{n^{\frac{1}{2}}}{2^n}$

Sol. $\Rightarrow a_n = \frac{n^{\frac{1}{2}}}{2^n}$

$\Rightarrow (a_n)^{\frac{1}{n}} = \left[\frac{(n)^{\frac{1}{2}}}{2^n}\right]^{\frac{1}{n}} = \frac{(n^{\frac{1}{2n}})}{2} = \frac{(n^{\frac{1}{n}})^{\frac{1}{2}}}{2}$

$\therefore \lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{(n^{\frac{1}{n}})^{\frac{1}{2}}}{2} = \frac{\sqrt{1}}{2}$

$= \boxed{\frac{1}{2} < 1}$ $\left(\because \lim_{n\to\infty} n^{\frac{1}{n}} = 1\right)$

So by Root Test $\sum a_n$ Converges

Note $\boxed{\lim_{n\to\infty} n^{\frac{1}{2n}} = 1}$ $\boxed{(\infty)^0}$

Let $y = n^{\frac{1}{2n}}$

$\Rightarrow \ln y = \frac{1}{2n} \ln n$

$\Rightarrow \lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{1}{2n} \ln n \; \boxed{(0\cdot\infty)}$

$= \lim_{n\to\infty} \frac{\ln n}{2n}$ $\left(\frac{\infty}{\infty}\right)$ L'Hospital

$= \frac{1/n}{2}$

$= 0$

$\lim_{n\to\infty} y = e^0$

$= 1$

Q.28 $\sum_1^\infty \frac{n^n \cdot 2^n}{(n+2)!}$

Sol. Here $a_n = \frac{n^n \cdot 2^n}{(n+2)!}$

Formulas –

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

$\Rightarrow a_{n+1} = \frac{(n+1)^{n+1} \cdot 2^{n+1}}{(n+3)!}$

$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1} \cdot 2^{n+1}}{(n+3)!} \times \frac{(n+2)!}{n^n \cdot 2^n}$

$= \frac{(n+1)(n+1)^n \cdot 2 \cdot \cancel{2^n} \times \cancel{(n+2)!}}{(n+3)\cancel{(n+2)!}\; n^n\, \cancel{2^n}}$

$= \frac{2(n+1)(n+1)^n}{(n+3)\,n^n} = \frac{2(n+1)}{n+3} \cdot \left(\frac{n+1}{n}\right)^n$

$\frac{a_{n+1}}{a_n} = \frac{2\left(1+\frac{1}{n}\right)}{\left(1+\frac{3}{n}\right)} \cdot \left(1+\frac{1}{n}\right)^n$

$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2\left(1+\frac{1}{n}\right)}{\left(1+\frac{3}{n}\right)} \cdot \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^\infty$

$= \boxed{2 \cdot e > 1}$ $\Rightarrow \sum a_n$ diverges.

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\left(2+\frac{3}{n}\right)\left(3+\frac{1}{3^n}\right)\left(1+\frac{1}{4^n}\right)}{\left(4+\frac{1}{4^n}\right)\left(2+\frac{1}{n}\right)\left(1+\frac{1}{3^n}\right)}$$

$$=\frac{2\cdot3\cdot1}{4\cdot2\cdot1}=\frac{6}{8}=\boxed{\frac{3}{4}<1}$$ Hence Convergent Series

Ratio **Q. 31** $\sum_{1}^{\infty}\frac{2^{2n-1}}{(2n-1)!}$

SoL. Here $a_n=\frac{2^{2n-1}}{(2n-1)!}$ & $a_{n+1}=\frac{2^{2n+1}}{(2n+1)!}$

Then $$\frac{a_{n+1}}{a_n}=\frac{2^{2n+1}}{(2n+1)!}\times\frac{(2n-1)!}{2^{2n-1}}$$

$$=\frac{2\cdot\cancel{2^{2n}}\times\cancel{(2n-1)!}}{(2n+1)\,2n\,\cancel{(2n-1)!}\,2^{-1}\,\cancel{2^{2n}}}=\frac{\cancel{2}\cdot2}{\cancel{2}n(2n+1)}=\frac{2}{n(2n+1)}$$

using ratio test,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{2}{n(2n+1)}=\boxed{0<1}$$ Hence Convergent Series

Ratio **Q. 32** $\sum_{1}^{\infty}\frac{n!}{e^{n^2}}$

SoL. Here $a_n=\frac{n!}{e^{n^2}}$ & $a_{n+1}=\frac{(n+1)!}{e^{(n+1)^2}}$

Then $$\frac{a_{n+1}}{a_n}=\frac{(n+1)!}{e^{(n+1)^2}}\times\frac{e^{n^2}}{n!}=\frac{(n+1)\cancel{n!}\cdot e^{n^2}}{e^{n^2+2n+1}\cdot\cancel{n!}}$$

$$\frac{a_{n+1}}{a_n}=\frac{(n+1)\cancel{(e^{n^2})}}{\cancel{(e^{n^2})}e^{2n+1}}$$

using ratio test, we get

Ratio **Q.29** $\displaystyle\sum_{1}^{\infty} \frac{(2n+1)!}{n^2(n+1)!}$

SoL. Here $a_n = \dfrac{(2n+1)!}{n^2(n+1)!}$

$$\Rightarrow a_{n+1} = \frac{(2n+3)!}{(n+1)^2(n+2)!}$$

then $\dfrac{a_{n+1}}{a_n} = \dfrac{(2n+3)!}{(n+1)^2(n+2)!} \times \dfrac{n^2(n+1)!}{(2n+1)!}$

$$\frac{a_{n+1}}{a_n} = \frac{(2n+3)(2n+2)\cancel{(2n+1)!}}{(n+1)^2(n+2)\cancel{(n+1)!}} \cdot \frac{n^2\,\cancel{(n+1)!}}{\cancel{(2n+1)!}}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^2(2n+3)(2n+2)}{(n+1)^2(n+2)} = \frac{n^4\left(2+\frac{3}{n}\right)\left(2+\frac{2}{n}\right)}{n^3\left(1+\frac{1}{n}\right)^2\left(1+\frac{2}{n}\right)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n\left(2+\frac{3}{n}\right)\left(2+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^2\left(1+\frac{2}{n}\right)} = \boxed{\infty > 1}$$

Hence Divergent series

Ratio **Q.30** $\displaystyle\sum_{1}^{\infty} \frac{(2n+1)(3^n+1)}{4^n+1}$

SoL. $a_n = \dfrac{(2n+1)(3^n+1)}{4^n+1}$

$$a_{n+1} = \frac{(2n+3)(3^{n+1}+1)}{(4^{n+1}+1)}$$

So $\dfrac{a_{n+1}}{a_n} = \dfrac{(2n+3)(3^{n+1}+1)}{(4^{n+1}+1)} \times \dfrac{(4^n+1)}{(2n+1)(3^n+1)}$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{n}\left(2+\frac{3}{n}\right)\cancel{3^n}\left(3+\frac{1}{3^n}\right)\cdot\cancel{4^n}\left(1+\frac{1}{4^n}\right)}{\cancel{4^n}\left(4+\frac{1}{4^n}\right)\cancel{n}\left(2+\frac{1}{n}\right)\cancel{3^n}\left(1+\frac{1}{3^n}\right)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2e^{2n+1}} = \frac{1}{\infty} = 0 < 1$$

$\Rightarrow \sum a_n$ is Convergent.

Integral **Q. 33** $\sum_{2}^{\infty} \frac{1}{\sqrt{n}(\ln n)^3}$

Sol Here $a_n = \frac{1}{\sqrt{n}(\ln n)^3}$

Let $b_n = \frac{1}{n \ln n}$

Then $\frac{a_n}{b_n} = \frac{n \ln n}{\sqrt{n}(\ln n)^3}$ $\Rightarrow$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{(\ln n)^2} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{2}{n}\ln n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{4\ln n} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{4\cdot\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{8}\sqrt{n} = \infty$$

but $\sum b_n = \sum \frac{1}{n \ln n}$ is divergent

$\because \int_2^{\infty} f(x)\,dx = \lim_{n\to\infty} \int_2^n \frac{1}{x \ln x}\,dx$

when $f(x) = \frac{1}{x \ln x}$

$$\int_2^{\infty} f(x)\,dx = $$

$$= \lim_{n\to\infty} \int_2^n \frac{\frac{1}{x}}{\ln x}\,dx$$

Q33 **ALTERNATE** LCT Easy

$a_n = \frac{1}{\sqrt{n}(\ln n)^3}$

Let $b_n = \frac{1}{n}$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}(\ln n)^3}\cdot\frac{n}{1}$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{(\ln n)^3} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{3(\ln n)^2}{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{6(\ln n)^2} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{12 \ln n}{n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{24 \ln n} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{24}{n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{48} = \infty$$

but $\sum b_n = \sum \frac{1}{n}$ is div.

So by L.C.T. given Series is div.

NOTE: We cannot take $b_n = \frac{1}{\sqrt{n}}$ because if so, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

So $\sum b_n$ should converge so that $\sum a_n$ cvgs but $\sum b_n = \sum \frac{1}{\sqrt{n}}$ dgs $\because P<1$

$$= \lim_{n\to\infty} \left|\ln(\ln n)\right|_2^n = \infty$$

$$\Rightarrow \sum b_n = \int_2^\infty f(x)\,dx \text{ is divergent}$$

So $\sum a_n$ is divergent.

Ratio **Q. 34** $\quad 1 + \frac{1\cdot 2}{1\cdot 3} + \frac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5} + \frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7} + \cdots$

Sol. nth term $a_n = \frac{1\cdot 2\cdot 3\cdot 4 \cdots n}{1\cdot 3\cdot 5\cdot 7 \cdots (2n-1)}$

$a=1,\ d=2,\ n=n$
$a_n = ?$
$a_n = a+(n-1)d$
$= 1+(n-1)2$
$= 2n-1$

$$a_{n+1} = \frac{1\cdot 2\cdot 3\cdot 4 \cdots n\cdot(n+1)}{1\cdot 3\cdot 5\cdot 7 \cdots (2n-1)(2n+1)}$$

Then $\frac{a_{n+1}}{a_n} = \frac{1\cdot 2\cdot 3\cdot 4 \cdots n(n+1)}{1\cdot 3\cdot 5\cdot 7 \cdots (2n-1)(2n+1)} \times \frac{1\cdot 3\cdot 5\cdot 7 \cdots (2n-1)}{1\cdot 2\cdot 3\cdot 4 \cdots n}$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2n+1} = \frac{\cancel{n}\left(1+\frac{1}{n}\right)}{\cancel{n}\left(2+\frac{1}{n}\right)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{2+\frac{1}{n}}\right) = \boxed{\frac{1}{2} < 1}$$ Hence Convergent Series

Ratio **Q. 35** $\quad \frac{2}{5} + \frac{2\cdot 4}{5\cdot 8} + \frac{2\cdot 4\cdot 6}{5\cdot 8\cdot 11} + \cdots$

$a=5,\ d=3,\ n=n$
$a_n = ?$
$a_n = a+(n-1)d$
$= 5+(n-1)3$
$= 3n+2$

Sol. $\Rightarrow a_n = \frac{2\cdot 4\cdot 6\cdot 8 \cdots 2n}{5\cdot 8\cdot 11 \cdots (3n+2)}$

$$\Rightarrow a_{n+1} = \frac{2\cdot 4\cdot 6\cdot 8 \cdots 2n\cdot(2n+2)}{5\cdot 8\cdot 11 \cdots (3n+2)(3n+5)}$$

Then $\frac{a_{n+1}}{a_n} = \frac{2\cdot 4\cdot 6\cdot 8 \cdots 2n(2n+2)}{5\cdot 8\cdot 11 \cdots (3n+2)(3n+5)} \times \frac{5\cdot 8\cdot 11 \cdots (3n+2)}{2\cdot 4\cdot 6\cdot 8 \cdots 2n}$

$$= \frac{2n+2}{3n+5} = \frac{\cancel{n}\left(2+\frac{2}{n}\right)}{\cancel{n}\left(3+\frac{5}{n}\right)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{2+\frac{2}{n}}{3+\frac{5}{n}}\right) = \boxed{\frac{2}{3} < 1}$$ So by ratio test $\sum a_n$ is Convergent

Q. 36 $\sum_{1}^{\infty} \frac{(a+1)(2a+1)(3a+1)\cdots(na+1)}{(b+1)(2b+1)(3b+1)\cdots(nb+1)}$, $a>0$, $b>0$

Sol. We have $a_n = \frac{(a+1)(2a+1)(3a+1)\cdots(na+1)}{(b+1)(2b+1)(3b+1)\cdots(nb+1)}$

$$a_{n+1} = \frac{(a+1)(2a+1)\cdots(na+1)\left[(n+1)a+1\right]}{(b+1)(2b+1)\cdots(nb+1)\left[(n+1)b+1\right]}$$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{(a+1)(2a+1)\cdots(na+1)\left[(n+1)a+1\right]}{(b+1)(2b+1)\cdots(nb+1)\left[(n+1)b+1\right]} \times \frac{(b+1)(2b+1)\cdots(nb+1)}{(a+1)(2a+1)\cdots(na+1)}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)a+1}{(n+1)b+1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{na+a+1}{nb+b+1} = \lim_{n\to\infty} \frac{\cancel{n}\left(a+\frac{a}{n}+\frac{1}{n}\right)}{\cancel{n}\left(b+\frac{b}{n}+\frac{1}{n}\right)}$$

$$= \frac{a+0+0}{b+0+0} = \frac{a}{b}$$

Then by ratio test

(i) $\sum a_n$ converges if $\frac{a}{b} < 1$

(ii) $\sum a_n$ diverges if $\frac{a}{b} > 1$

(iii) but if $\frac{a}{b} = 1$ then $\Rightarrow a = b$

Then $a_n = \frac{(a+1)}{(a+1)} \frac{(2a+1)}{(2a+1)} \frac{(3a+1)}{(3a+1)} \cdots \frac{(na+1)}{(na+1)}$

$\Rightarrow a_n = 1 \cdot 1 \cdot 1 \cdot 1 \cdots\cdots n$ times $= 1$

So apply div. test i.e

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 = 1 \neq 0$$

Thus by divergent test for $a = b$

Given series is divergent.

Q.37 If $x>0$, Show that the series $\sum_{1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$ Converges for $x<4$.

SoL: Here $a_n = \frac{(n!)^2}{(2n)!} x^n$

and $a_{n+1} = \frac{[(n+1)!]^2}{(2n+2)!} x^{n+1}$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2 \cancel{x^{n}} x}{(2n+2)!} \times \frac{(2n)!}{(n!)^2 \cancel{x^n}} = \frac{((n+1))^2 \cancel{(n!)^2} \times \cancel{(2n)!}\, x}{(2n+2)(2n+1)\cancel{2n!}\,\cancel{(n!)^2}}$$

$$= \frac{(n+1)^2 x}{(2n+2)(2n+1)} = \frac{\cancel{n^2}\left(1+\frac{1}{n}\right)^2 x}{\cancel{n^2}\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2 x}{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)} = \boxed{\frac{x}{4}}$$

Now according to ratio test

$\sum a_n$ will converges only if $\frac{x}{4}<1$. and it is only possible when $x<4$

Q.38 If $x>0$, Prove that series $\sum_{1}^{\infty} \frac{1\cdot3\cdot5\cdots(2n-1)}{1\cdot4\cdot7\cdots(3n-2)} x^n$ converges for $x<\frac{3}{2}$

SoL. We have $a_n = \frac{1\cdot3\cdot5\cdots(2n-1)}{1\cdot4\cdot7\cdots(3n-2)} x^n$

$$\Rightarrow a_{n+1} = \frac{1\cdot3\cdot5\cdots(2n-1)(2n+1)}{1\cdot4\cdot7\cdots(3n-2)(3n+1)} x^{n+1}$$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{\cancel{1\cdot3\cdot5\cdots(2n-1)}(2n+1)\, x\cdot\cancel{x^n}}{\cancel{1\cdot4\cdot7\cdots(3n-2)}(3n+1)} \times \frac{\cancel{1\cdot4\cdot7\cdots(3n-2)}}{\cancel{1\cdot3\cdot5\cdots(2n-1)\,x^n}}$$

$$\therefore \quad = \frac{2n+1}{3n+1} x = \frac{\cancel{n}\left(2+\frac{1}{n}\right)x}{\cancel{n}\left(3+\frac{1}{n}\right)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{\left(2+\frac{1}{n}\right)x}{\left(3+\frac{1}{n}\right)}\right) = \frac{2}{3}x$$

By Ratio test
Σa_n cgs if $\frac{2}{3}x < 1$

$\Rightarrow x < \frac{3}{2}$

Q.39 Show that $\sum_1^\infty \frac{x^n}{n!}$ converges $\forall$ +ive values of x.

Sol. $\because a_n = \frac{x^n}{n!}$ & $a_{n+1} = \frac{x^{n+1}}{(n+1)!}$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{x \cdot \cancel{x^n}}{(n+1)!} \times \frac{n!}{\cancel{x^n}} = \frac{x \cdot \cancel{n!}}{(n+1)\,\cancel{n!}} = \frac{x}{n+1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x}{n+1} = \boxed{0<}\ \forall \text{ +ive values of } x.$$

$\Rightarrow$ by ratio test Σa_n convergent $\forall$ +ive values of x.

Q.40 Find those +ive values of x for which the series $1 + \sum_1^\infty \frac{x^{2n}}{2^n}$ converges

Sol. It is obvious that Series will converges if $\sum_1^\infty \frac{x^{2n}}{2n}$ converges

$\because$ addition and substraction of finite number of terms does not affect the convergence or divergence of an infinite series.

So $a_n = \frac{x^{2n}}{2n}$

$$\Rightarrow a_{n+1} = \frac{x^{2n+2}}{2n+2} = \frac{x^2 \cdot x^{2n}}{2(n+1)}$$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{x^2 \cdot x^{2n}}{\cancel{2}(n+1)} \times \frac{2n}{\cancel{x^{2n}}} = \frac{x^2 \cdot n}{n(1+\frac{1}{n})} = \frac{x^2}{1+\frac{1}{n}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^2}{1+\frac{1}{n}} = x^2$$

Hence by ratio test the given series will converge if $x^2 < 1$

$x < 1$ (only +ive values of x)

$0 < x < 1$