Exercise 8.4 - Chapter 08
Mathematical Methods
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# ALTERNATING SERIES.

A Series in which terms are alternately +ive and negative (or negative and positive) is called an Alternating Series, Thus the series

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 \cdots\cdots = \sum_1^{\infty} (-1)^{n-1} a_n$$

and

$$-a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - \cdots\cdots = \sum_1^{\infty} (-1)^n a_n$$

where 'a' is positive, are alternating Series.

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# ABSOLUTE CONVERGENCE OF ALTERNATING SERIES

Let $\sum_1^{\infty} (-1)^{n-1} a_n$ be an alternating Series. Then if the series $\sum_1^{\infty} \left|(-1)^{n-1} a_n\right|$ is convergent, we say that the alternating Series is "Absolutely Convergent".

[illegible]

# CONDITIONALLY CONVERGENT

[illegible]

If the Series $\sum_1^{\infty} \left|(-1)^{n-1} a_n\right|$ is divergent but $\sum_1^{\infty} (-1)^{n-1} a_n$ is convergent, we say that the alternating series is Conditionally Convergent.

**EXAMPLE** The Series $\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{1}{n}\right|$ Converges conditionally but not absolutely.

**Sol** :- $\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{1}{n}\right| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ as $\sum_1^{\infty} \frac{1}{n}$ is divergent

Where as $\sum_1^{\infty} (-1)^{n-1} \frac{1}{n}$ is convergent

Since here $a_n = \frac{1}{n}$, $a_{n+1} = \frac{1}{n+1}$

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$\Rightarrow \quad n < n+1 \quad \forall$ +ive integral values of $n$

$\Rightarrow \quad \frac{1}{n} > \frac{1}{n+1} \quad \forall$ positive integral values of

$\Rightarrow \quad a_n > a_{n+1} \quad$ " " " "

Also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

So both conditions for an alternating series to be convergent are satisfied. Hence $\sum_1^\infty (-1)^{n-1}\frac{1}{n}$ is convergent. Hence proved.

∵ an alternating Series $a_1 - a_2 + a_3 - a_4 + \cdots = \sum (-1)^{n-1} a_n$ such that $a_n > 0$, $\forall\, n = 1, 2, 3, \ldots$ converge if
① $\lim_{n\to\infty} a_n = 0$
② $a_n > a_{n+1}$ $\forall\, n = 1, 2, 3, 4, \ldots$

## Theorem (Alternating Series test)

※ It is also called Leibniz test

Let $a_n > 0$, for $n = 1, 2, 3, 4, \ldots$, then the series

$$a_1 - a_2 + a_3 - a_4 + a_5 + \cdots + (-1)^{n-1} a_n + \cdots = \sum_1^\infty (-1)^{n-1} a_n$$

converges, if following two conditions are satisfied

(i) $\lim_{n\to\infty} a_n = 0$

(ii) $a_n > a_{n+1}$ for all +ve integral values of $n$ i.e $\{a_n\}$ is non increasing.

(Abutant is non increasing sequence if $f'(x) < 0$ & $\lim_{n\to\infty} a_n = 0$ see Q3,5)

**Note:-** If any one condition of alternating series test for convergence is not satisfied. Then given alternating series will be divergent.

**Proof** Let $S_n$ be the $n$th partial sum of the series $\sum (-1)^{n-1} a_n$ i.e $S_n = \sum (-1)^{n-1} a_n$

then $S_{2n} = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots - a_{2n-2} + a_{2n-1} - a_{2n}$

$$= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n}$$

$$S_{2n} = a_1 - \{(a_2 - a_3) + (a_4 - a_5) + \cdots + (a_{2n-2} - a_{2n-1})\} - a_{2n}$$

$\Rightarrow \quad S_{2n} < a_1 \quad (\because a_{n+1} < a_n \text{ and } a_n > 0 \ \forall n)$

$\Rightarrow$ Sequence $\{S_{2n}\}$ is bounded above

Now $S_{2n+2} = (a_1 - a_2 + a_3 - a_4 + \cdots - a_{2n}) + a_{2n+1} - a_{2n+2}$

$$S_{2n+2} = S_{2n} + a_{2n+1} - a_{2n+2}$$

$\Rightarrow \quad S_{2n+2} - S_{2n} = a_{2n+1} - a_{2n+2}$

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$\Rightarrow S_{2n+2} - S_{2n} > 0 \quad \forall n$ $\left(\begin{array}{l}\text{Since } a_n > 0 \ \forall n \\ \Rightarrow a_{2n+1} > 0,\ a_{2n+2} > 0 \\ \text{and } a_{2n+2} < a_{2n+1}\end{array}\right)$

$\Rightarrow S_{2n+2} > S_{2n}.$

$\Rightarrow \{S_{2n}\}$ the Sequence, is monotonically increasing

$\Rightarrow$ The sequence $\{S_{2n}\}$ is Convergent

let $\lim_{n\to\infty} S_{2n} = S$ (a finite real number, the according to def. of convergent sequence)

Now $S_{2n+1} = (a_1 - a_2 + a_3 - a_4 + \cdots - a_{2n}) + a_{2n+1}$

or $S_{2n+1} = S_{2n} + a_{2n+1}$

$\Rightarrow \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1}$

$= S + 0 = S$

$\Rightarrow$ The Sequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ converges to same real numbers. i.e.

$\lim_{n\to\infty} S_n = S$ both for even and odd values of $n$. Hence the given Series is Convergent.

a Sequence $\{a_n\}$ is monotonically increasing if $a_{n+1} > a_n \ \forall n$. i.e $a_1 < a_2 < a_3 < a_4 \cdots\cdots$

$\because$ a Sequence which is bounded above and monotonically increasing is Convergent

$\because$ if finite number of terms in a infinite Series does not effect its behaviour

$\because \lim_{n\to\infty} a_n = 0$

$\Rightarrow \lim_{n\to\infty} a_{2n+1} = 0$

**Theorem:-** If the series $\sum_1^\infty |a_n|$ Converges then so does the series i.e, if a series converges absolutely then it Converges.

**Proof:-** Let $\sum_1^\infty |a_n| = |a_1| + |a_2| + |a_3| + \cdots + |a_n| + \cdots$

be convergent.

Consider the series

$$\sum_1^\infty (a_n + |a_n|)$$

We have $a_n + |a_n| = \begin{cases} 0, & \text{if } a_n \text{ is negative} \\ 2|a_n|, & \text{if } a_n \text{ is positive.}\end{cases}$

Hence $0 \le a_n + |a_n| \le 2|a_n| \quad \longrightarrow (1)$

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## Theorem (Root Test for Absolute Convergent)

Let $\sum_{1}^{\infty} a_n$ be an infinite Series and $\lim_{n\to\infty} (|a_n|)^{\frac{1}{n}} = l$, where $l$ is non-negative number or $\infty$.

(i) if $l<1$. The Series is absolutely convergent.

(ii) if $l>1$ or $\infty$, The Series is divergent.

(iii) If $l=1$, the test fails and the series may be absolutely convergent, conditionally convergent or divergent.

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# Exercise 8.4

use the alternating Series test to determine whether the given Series converges (Problems 1–6):

**ALTERNATING SERIES TEST (i.e. Laibniz Test)**

The alternating Series $\sum(-1)^{n-1} a_n$ is convergent if

(i) $a_n > a_{n+1}$ i.e. nonincreasing seq $\{a_n\}$

(ii) $\lim_{n\to\infty} a_n = 0$

(1) $\sum_{2}^{\infty} (-1)^n \frac{1}{n\ Ln\, n}$

**Sol.** Here $a_n = \frac{1}{n\ Ln\, n}$ ($n^{th}$ term) $\Rightarrow a_{n+1} = \frac{1}{(n+1)\, Ln(n+1)}$

$\because\ n\, Ln\, n < (n+1)\, Ln(n+1)$

$\Rightarrow \frac{1}{n\, Ln\, n} > \frac{1}{(n+1)\, Ln(n+1)}$

$\Rightarrow a_n > a_{n+1}$ so $\{a_n\}$ is nonincreasing seq.

$\{a_n\} = \left\{\frac{1}{n\, Ln\, n}\right\}$

Alternative:
$f(x) = \frac{1}{x\ln x} = (x\ln x)^{-1}$
$f'(x) = -(x\ln x)^{-2}\left\{x\cdot\frac{1}{x} + \ln x\right\} < 0$
$\therefore$ Seq $\{a_n\}$ is non increasing

Also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n\, Ln\, n} = 0$

So by Alternating Series test. $\sum_{2}^{\infty}(-1)^n \frac{1}{n\, Ln\, n}$ is convergent. (Since both conditions of convergence of an alternating series are satisfied)

## Q.2 $\sum_{1}^{\infty} \frac{\cos n\pi}{\sqrt{n\pi}}$

**Sol.** Here $\cos n\pi = \begin{cases} -1, & \text{when } n \text{ is odd} \\ 1, & \text{when } n \text{ is even} \end{cases}$

$\cos n\pi = -1+1-1+1-\cdots = (-1)^n$

$$\therefore \sum_{1}^{\infty} \frac{\cos n\pi}{\sqrt{n\pi}} = \sum_{1}^{\infty} \frac{(-1)^n}{\sqrt{n\pi}} \quad \text{(Alternating series)}$$

$$\Rightarrow a_n = \frac{1}{\sqrt{n\pi}} \quad \Rightarrow \quad a_{n+1} = \frac{1}{(n+1)\pi}$$

Since $n\pi < (n+1)\pi$

$$\Rightarrow \frac{1}{n\pi} > \frac{1}{(n+1)\pi} \quad \Rightarrow \quad \frac{1}{\sqrt{n\pi}} > \frac{1}{\sqrt{(n+1)\pi}}$$

$\Rightarrow a_n > a_{n+1}$ $\therefore$ $\{a_n\}$ is non increasing seq.

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n\pi}} = 0$

So by alternating series test, the given series is convergent.

2nd Method

$f(x) = \frac{1}{\sqrt{x\pi}} = (x\pi)^{-1/2}$

$f'(x) = -\frac{1}{2}(x\pi)^{-3/2}(\pi) = \frac{-\pi}{2(x\pi)^{3/2}} < 0$

this seq $\{a_n\}$ is non-increasing

## Q.3 $\sum_{1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1}$

**Sol** Here $a_n = \frac{n^2}{n^2+1}$, $\Rightarrow$ $a_{n+1} = \frac{(n+1)^2}{(n+1)^2+1}$ (No need of it)

Since second condition is not satisfied i.e. $\lim_{n\to\infty} a_n \neq 0$ so is not convergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^2+1} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1 \neq 0$$

This cond is not satisfied. So is divergent. So Series is not convergent.

2nd Method

$f(x) = \frac{x^2}{x^2+1}$

$f'(x) = \frac{(x^2+1)2x - x^2(2x)}{(x^2+1)^2}$

$f'(x) = \frac{2x}{(x^2+1)^2} > 0$

$\Rightarrow \{a_n\} = \left\{\frac{n^2}{n^2+1}\right\}$ is increasing not non-increasing

and $\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0$

$\therefore$ Not Cgt

$x \to \infty$

## Q.4 $\sum_{1}^{\infty} (-1)^{n-1} \frac{1}{e^n}$

**Sol.** Here $a_n = \frac{1}{e^n}$ and $a_{n+1} = \frac{1}{e^{n+1}}$

$\because$ Since $e^n < e^{n+1}$

$\Rightarrow \dfrac{1}{e^n} > \dfrac{1}{e^{n+1}}$ i.e $a_n > a_{n+1}$.

also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{1}{e^n} = 0$

$\therefore$ by alternating series test, $\sum_1^\infty (-1)^{n-1}\dfrac{1}{e^n}$ is Convergent.

2nd Method

$f(x) = \dfrac{1}{e^x} = e^{-x}$

$f'(x) = -\dfrac{1}{e^x} < 0$

Seq is non increasing

## Q.5 $\sum_1^\infty (-1)^{n-1}\dfrac{n+4}{n^2+n}$

**Sol:** Here $a_n = \dfrac{n+4}{n^2+n} \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}\dfrac{n+4}{n^2+n} \quad \left(\dfrac{\infty}{\infty}\right)$

or $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\dfrac{n\left(1+\frac{4}{n}\right)}{n^2\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty}\dfrac{1+\frac{4}{n}}{n\left(1+\frac{1}{n}\right)} = \boxed{0}$

$a_{n+1} = \dfrac{n+1+4}{(n+1)^2+(n+1)} = \dfrac{n+5}{(n+1)(n+1+1)} = \dfrac{n+5}{(n+1)(n+2)}$

$\therefore a_n - a_{n+1} = \dfrac{n+4}{n(n+1)} - \dfrac{n+5}{(n+1)(n+2)}$

$= \dfrac{(n+2)(n+4) - n(n+5)}{n(n+1)(n+2)}$

$a_n - a_{n+1} = \dfrac{n^2+6n+8-n^2-5n}{n(n+1)(n+2)} = \dfrac{n+8}{n(n+1)(n+2)} \quad (1)$

2nd Method

$f(n) = \dfrac{n+4}{n^2+n}$

$f'(n) = \dfrac{(n^2+n)(1)-(n+4)(2n+1)}{(n^2+n)^2}$

$= \dfrac{n^2+n-2n^2-8n-n-4}{(n^2+n)^2}$

$= \dfrac{-n^2-8n-4}{(n^2+n)^2} < 0$

$\Rightarrow \{a_n\}$ is non-increasing

ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\dfrac{n+4}{n^2+n} = 0$

$\therefore$ Series is cgt.

Since (1) is a +ive quantity, because $n$ is any +ive integer.

$\Rightarrow a_n - a_{n+1} > 0 \Rightarrow a_n > a_{n+1}$.

Since both conditions for an alternating series to be convergent. Thus given Series is convergent.

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V.Imp Note:
If Mod i.e $\sum |a_n|$ cgs then Al. Series i.e $\sum a_n$ is Ab. Cgt
If Mod i.e $\sum |a_n|$ dgs but Al. Series i.e $\sum a_n$ Cgs (using AST) then conditional Cgt
If Mod dgs and Al.S. $\sum a_n$ also dgs then total Dgt

**Q.7** $\sum_{1}^{\infty} \frac{(-1)^n\, n!}{2n!}$

**SoL.** $a_n = \frac{(-1)^n\, n!}{(2n)!}$

$$\Rightarrow |a_n| = \frac{n!}{2n!} \quad \text{and} \quad |a_{n+1}| = \frac{(n+1)!}{(2n+2)!}$$

using ratio test for absolute convergence test

$$\text{i.e } \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)!}{(2n+2)!} \times \frac{2n!}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\,\cancel{n!}}{(2n+2)(2n+1)\cancel{2n!}} \times \frac{\cancel{2n!}}{\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{1}{n}\right)}{n^{2}\left[2+\frac{2}{n}\right]\left[2+\frac{1}{n}\right]} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{n\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)} = 0 < 1$$

So by Ratio test for (absolute convergence) the given Series is absolutely convergent.

**Q.8** $\sum_{1}^{\infty} \frac{\sin\sqrt{n}}{\sqrt{n^3+1}}$

**SoL.** Here Mod $|a_n| = \left|\frac{\sin\sqrt{n}}{\sqrt{n^3+1}}\right|$.

Now

$\sin\sqrt{n} \leq 1$

$\frac{\sin\sqrt{n}}{\sqrt{n^3+1}} \leq \frac{1}{\sqrt{n^3+1}}$

$\frac{\sin\sqrt{n}}{\sqrt{n^3+1}} \leq \frac{1}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3}}$

$\because$ values of $|\sin\sqrt{n}|$ lies between $[0,1]$

but $\frac{1}{n^{3/2}}$ is cgt

$(\because p = 3/2 > 1)$ So

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So by B.C. Test $\sum a_n$ is absolutely convergent

(An absolute cgt series is convergent)

ab. cgt $\Rightarrow$ cgt Mod

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$$Q.6 \quad \sum_{1}^{\infty} (-1)^n \frac{\ln n}{n}$$

2nd Method

$f(x) = \frac{\ln n}{n}$

$f'(x) = \frac{n \frac{1}{n} - \ln n}{n^2}$

$= \frac{1-\ln n}{n^2} < 0$ when $n > 3$

$\Rightarrow \left\{\frac{\ln n}{n}\right\}$ is non-increasing

(ii) $\lim_{n\to\infty} \frac{\ln n}{n}$

$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1}$ (L'Hospital Rule)

$= 0$

Sol

Here $a_n = \frac{\ln n}{n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} \quad \left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1}$ (L'Hospital Rule)

$\lim_{n\to\infty} a_n = 0$

& $a_{n+1} = \frac{\ln(n+1)}{n+1}$

$\therefore a_n - a_{n+1} = \frac{\ln n}{n} - \frac{\ln(n+1)}{n+1}$

$= \frac{(n+1)\ln n - n\ln(n+1)}{n(n+1)}$

$= \frac{n\ln n + \ln n - n\ln(n+1)}{n(n+1)}$

$= \frac{n\ln n - n\ln(n+1) + \ln n}{n(n+1)}$

$= \frac{n(\ln n - \ln(n+1)) + \ln n}{n(n+1)}$

$a_n - a_{n+1} = \frac{n\ln\left(\frac{n}{n+1}\right) + \ln n}{n(n+1)} > 0$ ($\because n$ is any +ive integer)

ALTERNATE :

To prove $a_n > a_{n+1}$

consider $\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)}{n+1} \times \frac{n}{\ln n} < 1$

$\forall \; n \in Z^+$

(i.e +ive integral values of $n$)

$\Rightarrow \frac{a_{n+1}}{a_n} < 1 \Rightarrow a_{n+1} < a_n \;\; \forall\, n \in Z^+$

$\Rightarrow a_n > a_{n+1}$. Thus by alternating series test

$\Rightarrow$ The given series is convergent.

Test the given series for

(i) absolute convergent (ii) Conditionally convergent

(iii) Divergent

(PROBLEMS 7—24)

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**Q. 9** $\sum_{1}^{\infty} \frac{(-1)^n (n+2)}{n(n+1)}$ $\Rightarrow$ $a_n = \frac{(-1)^n (n+2)}{n(n+1)}$

step1. **SOL.** from given Series $|a_n| = \frac{n+2}{n(n+1)} = \left|\frac{(-1)^n (n+2)}{n(n+1)}\right|$

$\Rightarrow |a_{n+1}| = \frac{n+3}{(n+1)(n+2)}$

using ratio test for absolutely convergent

i.e $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n+3}{(\cancel{n+1})(n+2)} \times \frac{n(\cancel{n+1})}{(n+2)} = \lim_{n\to\infty} \frac{\cancel{n^2}(1+\frac{3}{n})}{\cancel{n^2}(1+\frac{2}{n})(1+\frac{2}{n})} = 1$

$\Rightarrow$ Ratio test fails to determine that Alternating Series is Absolutely convergent or Conditionally convergent or divergent.

step3. $\therefore$ Now using Alternating series test

$\because |a_n| - |a_{n+1}| = \frac{n+2}{n(n+1)} - \frac{n+3}{(n+1)(n+2)}$

$= \frac{\cancel{n^2}+4n+4-\cancel{n^2}-3n}{n(n+1)(n+2)}$

$= \frac{n+4}{n(n+1)(n+2)} > 0$ for $n = 1, 2, 3, \ldots$

**ALTERNATING SERIES TEST** The alternating Series $(-1)^{n-1} a_n$ is convergent if (i) $a_n > a_{n+1}$ $\forall n = 1, 2, 3, \ldots$ and (ii) $\lim_{n\to\infty} a_n = 0$

$\Rightarrow |a_n| - |a_{n+1}| > 0$ i.e $|a_n| > |a_{n+1}|$

and $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n+2}{n(n+1)} = \lim_{n\to\infty} \frac{n(1+\frac{2}{n})}{n^2(1+\frac{1}{n})} = \lim_{n\to\infty} \frac{(1+\frac{2}{n})}{n(1+\frac{1}{n})} = 0$

So by Alternating Series test $\sum_{1}^{\infty} (-1)^n \frac{n+2}{n(n+1)}$ is Convergent.

step2. $\sum_{1}^{\infty} |a_n| = ?$ $\frac{(n+2)}{(n+1)} \cdot \frac{1}{n} > \frac{1}{n}$ $\frac{1}{n} = b_n$ (say)

then $|a_n| > |b_n|$ $\forall n = 1, 2, 3, \ldots$

& $\sum_{1}^{\infty} |b_n| = \sum_{1}^{\infty} \frac{1}{n}$ is divergent. ($\because$ Harmonic series)

$\therefore$ By C. Test $\Rightarrow \sum_{1}^{\infty} |a_n|$ is also Divergent. So check Alternating Series Test

$\therefore$ Alternating Series is Conditionally Convergent!

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**Q.10** $\sum_{1}^{\infty} (-1)^n \dfrac{n!}{1.3.5\ldots(2n-1)}$

**SoL.** $\Rightarrow a_n = \dfrac{(-1)^n n!}{1.3.5\ldots(2n-1)}$

So $|a_n| = \dfrac{n!}{1.3.5\ldots(2n-1)} \Rightarrow |a_{n+1}| = \dfrac{(n+1)!}{1.3.5\ldots\ldots(2n+1)}$

using ratio test for absolute convergent i.e

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)!^{\,n+1}}{1.3.5\ldots\cancel{(2n-1)}(2n+1)} \times \frac{1.3.5\ldots\cancel{(2n-1)}}{n!}$$

$$= \lim_{n\to\infty} \frac{n+1}{2n+1} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2+\frac{1}{n}} = \boxed{\frac{1}{2} < 1}$$

Hence Ab. Cgt Series i.e $\sum a_n$ is Ab cgt (using Ratio test for Ab cgt.)

**Q.11** $\sum_{1}^{\infty} (-1)^{n-1} \left(\dfrac{n+2}{3n-1}\right)^n$

**SoL.** Here $a_n = (-1)^{n-1} \left(\dfrac{n+2}{3n-1}\right)^n$

$\Rightarrow |a_n| = \dfrac{(n+2)^n}{(3n-1)^n}$

using root test for absolute convergent

i.e $$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \left(\left(\frac{n+2}{3n-1}\right)^n\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{2}{n}\right)}{\cancel{n}\left(3-\frac{1}{n}\right)} = \boxed{\frac{1}{3} < 1}$$

Hence Ab. Cgt Series

**Q.12** $\sum_{1}^{\infty} (-1)^n\, n \tan \frac{1}{n}$

**SoL.** $\Rightarrow a_n = (-1)^n\, n \tan \frac{1}{n}$

$\Rightarrow |a_n| = n \tan\left(\frac{1}{n}\right)$

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Dgt Test
1. (If $\underset{n\to\infty}{Lt}\ a_n \neq 0$ then $\Sigma a_n$ is dgt)

$$\underset{n\to\infty}{Lt}|a_n| = \underset{n\to\infty}{Lt}\ n\tan\frac{1}{n} = \underset{n\to\infty}{Lt}\ \frac{\tan(\frac{1}{n})}{1/n}$$

$$= \underset{n\to\infty}{Lt}\left(\left(\frac{\sin(\frac{1}{n})}{\frac{1}{n}}\right)\underset{n\to\infty}{Lt}\frac{1}{\cos(\frac{1}{n})}\right) = \boxed{1 \neq 0}$$ Hence $\sum_1^{\infty}|a_n|$ is dgt (i)

Now <u>Alternate Series Test</u> $\{a_n\} = \{n\tan\frac{1}{n}\}$

$\underset{n\to\infty}{Lt}\ a_n = \underset{n\to\infty}{Lt}\ n\tan\frac{1}{n} = 1 \neq 0$ (as above)

By Al. Series Test, $\Sigma a_n$ diverges as one cond is not satisfied (ii) of Al. Series Test

Therefore from (i) & (ii) $\Sigma a_n$ is <u>Totally divergent</u>

<u>Note</u> If Mod of Al-Series is proved dgt by Dgt Test i.e. $\underset{n\to\infty}{Lt}\ a_n \neq 0$ Then the series must be totally dgt. see Q12

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EoT 2001 A **Q13** $(-1)^{n-1}\frac{n^2}{(n+2)!}$

**Sol**

$$\Rightarrow a_n = \frac{(-1)^{n-1}n^2}{(n+2)!} \quad \text{So} \quad |a_n| = \frac{n^2}{(n+2)!}$$

Then $|a_{n+1}| = \frac{(n+1)^2}{(n+3)!}$

using ratio test

i.e. $$\underset{n\to\infty}{Lt}\frac{|a_{n+1}|}{|a_n|} = \underset{n\to\infty}{Lt}\frac{(n+1)^2}{(n+3)!}\times\frac{(n+2)!}{(n)^2}$$

$$= \underset{n\to\infty}{Lt}\frac{(n+1)^2}{(n+3)\cancel{(n+2)!}}\times\frac{\cancel{(n+2)!}}{(n)^2}$$

$$= \underset{n\to\infty}{Lt}\left(\frac{n+1}{n}\right)^2\cdot\frac{1}{(n+3)} = \underset{n\to\infty}{Lt}\left[\left(1+\frac{1}{n}\right)^2\cdot\frac{1}{n+3}\right] = \boxed{0<1}$$

So by ratio test, the given series is **Absolutely convergent.**

## Q.14 $\sum_1^{\infty}(-1)^{n-1}\left[\sqrt{n+1}-\sqrt{n}\right]$

**Sol.** $\Rightarrow a_n = (-1)^{n-1}\left[\sqrt{n+1}-\sqrt{n}\right] \Rightarrow |a_n| = \sqrt{n+1}-\sqrt{n}$ (by Rationalising)

$$= \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$\Rightarrow |a_{n+1}| = \frac{1}{\sqrt{n+2}+\sqrt{n+1}}$$

$$\underset{n\to\infty}{Lt}\frac{|a_{n+1}|}{|a_n|} = \underset{n\to\infty}{Lt}\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n+1}} = 1$$

اگر Rationalise نہ کریں تو
صورت میں
$\frac{|a_{n+1}|}{|a_n|} = \frac{\infty-\infty}{\infty-\infty} = \frac{0}{0}$
نتائج ہو گی L'Hospital کی ...

$\Rightarrow$ ratio test fails. So the given alternating Series is absolutely convergent or conditionally convergent or divergent.

Now Since $\sqrt{n} < \sqrt{n+1}$

$$\Rightarrow \quad \sqrt{n+1} + \sqrt{n} < \sqrt{n+1} + \sqrt{n+1} = 2\sqrt{n+1}$$

$$\Rightarrow \quad |a_n| = \frac{1}{\sqrt{n+1}+\sqrt{n}} > \frac{1}{2\sqrt{n+1}} = |b_n| \quad \forall\, n = 1, 2, 3, \cdots$$

but $|b_n| = \dfrac{1}{2\sqrt{n+1}}$ is divergent.

because $$\int_1^\infty \frac{1}{2(x+1)^{\frac{1}{2}}}\,dx = \frac{1}{2}\int_1^\infty (x+1)^{-\frac{1}{2}}\,dx = \frac{1}{2}\Big|(x+1)^{\frac{1}{2}}\Big|_1^\infty = \infty.$$

**So by Comparison test $\sum_1^\infty |a_n|$ is <u>divergent</u>.** (dgt کی تو dgt ہوگی)

<u>Now Al.S.T</u> $\quad \{a_n\} = \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n+1}} < 1 \quad \forall\, n = 1, 2, \ldots$$

$\Rightarrow |a_{n+1}| < |a_n| \quad \therefore \{a_n\}$ is decreasing

And $\displaystyle \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = \boxed{0}$

$f(x) = \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$

$f(x) = \sqrt{n+1} - \sqrt{n}$ after Rationalizing

$f'(x) = \frac{1}{2}(n+1)^{-\frac{1}{2}} - \frac{1}{2}(n)^{-\frac{1}{2}}$

$= \frac{1}{2}\left\{\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}\right\} < 0$

$\Rightarrow f(n)$ is dec $\Rightarrow \{a_n\}$ is dec

$\Rightarrow \Sigma(-1)^{n-1}(\sqrt{n+1}-\sqrt{n})$ is cgt.

So the given series is convergent (by A. Series. Test)

but $\sum_1^\infty |a_n|$ is divergent. $\Rightarrow$ Given Series i.e

$\Rightarrow \sum_1^\infty (-1)^{n-1}(\sqrt{n+1}-\sqrt{n})$ is conditionally convergent.

<u>Q. 15</u> $\quad \displaystyle\sum_2^\infty (-1)^n \frac{1}{\ln(\ln n)}$

<u>Sol.</u> $\Rightarrow a_n = (-1)^n \dfrac{1}{\ln(\ln n)}$, so $|a_n| = \dfrac{1}{\ln(\ln n)}$

Now Since $\ln n < n$

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$\Rightarrow \ln(\ln n) < \ln n$

Combining both inequalities

$\Rightarrow \ln(\ln n) < \ln n < n$

$\Rightarrow \dfrac{1}{\ln(\ln n)} > \dfrac{1}{n} = |b_n| \text{(say)} \Rightarrow |a_n| > |b_n|$

but $\sum_2^{\infty}|b_n| = \sum \frac{1}{n}$ is divergent Series

So $\sum_2^{\infty}|a_n|$ is Divergent $\longrightarrow$ ① چونکہ dgt بڑی تو dgt

Now Al.S.T

say $\{a_n\} = \left\{\frac{1}{\ln \ln n}\right\}$ $\quad |a_n| = \dfrac{1}{\ln(\ln n)} \Rightarrow |a_{n+1}| = \dfrac{1}{\ln(\ln(n+1))}$

$\Rightarrow \dfrac{|a_{n+1}|}{|a_n|} = \dfrac{\ln(\ln n)}{\ln(\ln(n+1))} < 1$

$\boxed{\begin{array}{l} \ln n < \ln(n+1) \\ \Rightarrow \ln(\ln n) < \ln(\ln(n+1)) \\ \Rightarrow \dfrac{\ln(\ln n)}{\ln(\ln(n+1))} < 1 \end{array}}$

$\Rightarrow |a_{n+1}| < |a_n| \quad \therefore \{a_n\}$ is non increasing seq.

also $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\dfrac{1}{\ln(\ln n)} = \boxed{0}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\dfrac{1}{\ln(\ln n)}$

$f(x) = \dfrac{1}{\ln(\ln x)} = (\ln(\ln x))^{-1}$

$f'(x) = -(\ln(\ln x))^{-2}\left(\dfrac{1}{x\ln x}\right) < 0$

$\Rightarrow \{a_n\}$ is decreasing

so by Al. Series Test $\sum_2^{\infty}(-1)^n\frac{1}{\ln \ln n}$ is cgt —— ②

combining ① & ② given Series is Conditionally cgt.

Q.16 $\sum_0^{\infty}(-1)^{n+1}\text{arc}\tan n = \sum_0^{\infty}(-1)^{n+1}\tan^{-1} n$

Sol. Here $a_n = (-1)^{n+1}\tan^{-1} n$

$\Rightarrow |a_n| = \tan^{-1} n$

Since $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\tan^{-1} n = \frac{\pi}{2}\boxed{\neq 0}$ so by divergent test

$\sum_1^{\infty}|a_n|$ is divergent $\longrightarrow$ ①

**A.S.T**

$\{a_n\} = \{\tan^{-1} n\}$

$\because \lim_{n\to\infty} a_n = \lim_{n\to\infty} \tan^{-1}(n)$

$= \frac{\pi}{2} \neq 0$

$\therefore \sum a_n$ dgs

$$\int_1^\infty f(x)\,dx = \lim_{n\to\infty} \int_1^n 1\cdot\tan^{-1}x\,dx$$

$$= \lim_{n\to\infty} \left\{ \left| x\tan^{-1}x \right| - \int \frac{x}{1+x^2}\,dx \right\}$$

$\because \frac{1}{2}\int \frac{2x}{1+x^2}dx = \ln(1+x^2)$

$$= \lim_{n\to\infty} \left\{ \left| x\tan^{-1}x \right| - \frac{1}{2} \left| \ln(1+x^2) \right|_1^n \right\}$$

$$= \lim_{n\to\infty} \left( n\tan^{-1}n - \tan^{-1}1 - \frac{1}{2}\left(\ln(1+n^2)\right) + \frac{1}{2}\ln 2 \right) = \infty$$

So by integral test, $\sum_1^\infty |a_n|$ is divergent $\rightarrow$ (2)

Combining statements (1) and (2), the given series is **Divergent**

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Integral

**Q.17** $\displaystyle\sum_2^\infty (-1)^{n-1} \frac{1}{n(\ln n)^2}$

$f(x) = \frac{1}{x(\ln x)^2} = x^{-1}(\ln x)^{-2}$

$f'(x) = -x^{-2}(\ln x)^{-2} - 2x^{-1}(\ln x)^{-3}\frac{1}{x}$

$= -\left[\frac{1}{x^{2}(\ln x)^{2}} + \frac{2}{x^{2}(\ln x)^3}\right]$

and $f'(x) < 0$.

$f(x)$ is non increasing for $x \geq 3$

**Sol:** $\Rightarrow |a_n| = \frac{1}{n(\ln n)^2}$

$\because f(x)$ is non increasing

So by integral test

$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{n\to\infty} \int_2^n (\ln x)^{-2}\left(\frac{1}{x}\right)dx$$

$$= \lim_{n\to\infty} \left| \frac{(\ln x)^{-1}}{-1} \right|_2^n = \lim_{n\to\infty}\left(\frac{-1}{\ln n} + \frac{1}{\ln 2}\right)$$

$$= \frac{1}{\ln 2}$$

$\Rightarrow \sum_2^\infty |a_n|$ is convergent (by Integral test)

So by definition of absolute convergent of an alternating series. The given series is **Absolutely Convergent**.

Imp CC 2009 A **Q.18** $\sum_{1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{n+1}$

**Sol.** Here $|a_n| = \frac{\sqrt{n}}{n+1}$ , $b_n = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n+1} \cdot \frac{\sqrt{n}}{1}$$

$$= \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})} = 1 \neq 0$$

$\therefore$ both behaves alike

$\frac{1}{2} - 1 = -\frac{1}{2}$

but $\sum_{1}^{\infty} b_n = \sum \frac{1}{\sqrt{n}}$ is divergent

So by L.C.T

$\sum_{1}^{\infty} |a_n|$ diverges ——— ①

Now A.S.T

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+1}$$

$\{a_n\} = \{\frac{\sqrt{n}}{n+1}\}$

$$= \lim_{n\to\infty} \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}(\sqrt{n}+\frac{1}{\sqrt{n}})}$$

$$\boxed{\lim_{n\to\infty} a_n = \boxed{0}}$$

$\because$ A Sequence $\{a_n\}$ is said to be non-increasing if for $a_n = f(n)$ and $f'(n) < 0$

it means. if a sequence $\{a_n\}$ is non increasing i.e $a_n \geq a_{n+1}$ for $n = 1, 2, 3, \ldots$ Then it ...

$\because$ $f(n) = \frac{\sqrt{n}}{n+1}$

$$f'(n) = \frac{\frac{1}{2\sqrt{n}}(n+1) - \sqrt{n}}{(n+1)^2}$$

$$= \frac{n+1-2n}{2\sqrt{n}(n+1)^2} = \frac{1-n}{2\sqrt{n}(n+1)^2} \boxed{<0} \text{ for } n \geq 2$$

$\Rightarrow$ $\{a_n\}$ is non increasing Sequence i.e $a_1 \geq a_2 \geq a_3 \geq a_4 \ldots\ldots$

So by alternating Series test when $\{a_n\}$ is non increasing sequence and $\lim_{n\to\infty} a_n = 0$. Then the alternating Series converges ---→ ②

So from statement ① & ②, We concluded that the given series Conditionally Convergent

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**NOTE** a sequence $\{a_n\}$ is also Non Increasing if

$$\frac{|a_{n+1}|}{|a_n|} < 1 \quad \text{i.e } |a_{n+1}| < |a_n|$$
$$\text{or } |a_n| > |a_{n+1}|$$

e.g if $a_n = \dfrac{\sqrt{n}}{n+1} \Rightarrow a_{n+1} = \dfrac{\sqrt{n+1}}{n+2}$

then $$\frac{a_{n+1}}{a_n} = \frac{\sqrt{n+1}}{n+2} \times \frac{n+1}{\sqrt{n}} = \sqrt{\frac{n+1}{n}} \cdot \frac{n+1}{n+2}$$

$$= \sqrt{1+\frac{1}{n}} \times \frac{n+1}{n+2} < 1 \quad \text{for all } n = 1, 2, 3, \ldots$$

**Q.19** $\displaystyle\sum_{1}^{\infty} \frac{(-1)^{n-1} n^2}{(2n+1)(n+5)}$

**SoL.** $\because |a_n| = \dfrac{n^2}{(2n+1)(n+5)}$ and $|a_{n+1}| = \dfrac{(n+1)^2}{(2n+3)(n+6)}$

using ratio test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+3)(n+6)} \times \frac{(2n+1)(n+5)}{n^2}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}(1+\frac{1}{n})^2}{\cancel{n^2}\left[2+\frac{3}{n}\right]\left[1+\frac{6}{n}\right]} \times \frac{\cancel{n^2}\left[2+\frac{1}{n}\right]\left[1+\frac{5}{n}\right]}{\cancel{n^2}} = 1.$$

i.e no conclusion can be drawn

Now divergent test

i.e $$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^2}{(2n+1)(n+5)} = \lim_{n\to\infty} \frac{1}{(2+\frac{1}{n})(1+\frac{5}{n})}$$

$$= \boxed{\frac{1}{2} \neq 0} \Rightarrow \sum_{1}^{\infty} |a_n| \text{ is divergent.}$$

Divergent test
If $\lim_{n\to\infty} a_n \neq 0$
Then $\sum a_n$ is divergent

A.S.Test $\{a_n\} = \left\{\dfrac{n^2}{(2n+1)(n+5)}\right\}$

$$\lim_{n\to\infty} a_n = \lim \frac{n^2}{(2n+1)(n+5)} = \frac{1}{2} \neq 0$$

$\therefore \sum a_n$ dgs.

[illegible]

or $f'(n) = \dfrac{\cancel{4n^3} + 22n^2 + 10n - \cancel{4n^3} - 11n^2}{(2n^2+11n+5)^2}$

$= \dfrac{11n^2+10n}{(2n^2+11n+5)^2} > 0$ for all $n = 1, 2, 3, \cdots$

$\Rightarrow$ $f(n)$ is increasing i.e sequence $\{a_n\}$ is increasing sequence.

So both conditions of alternating series test are not satisfied. Thus Given series is not convergent. Implies that the given series is divergent.

**Q.20** $\displaystyle\sum_{1}^{\infty} (-1)^{n-1} \frac{2\sinh n}{e^{2n}}$

**SoL.** Here $|a_n| = \dfrac{2\sinh n}{e^{2n}} = \dfrac{\cancel{2}\left(\dfrac{e^n - e^{-n}}{\cancel{2}}\right)}{e^{2n}}$

$$|a_n| = \frac{e^n - \frac{1}{e^n}}{e^{2n}} = \frac{e^{2n}-1}{e^{3n}} < \frac{e^{2n}}{e^{3n}}$$

$$\therefore \Rightarrow |a_n| = \frac{e^{2n-1}}{e^{3n}} < \frac{1}{e^n}$$

Let $\Sigma b_n = \Sigma \dfrac{1}{e^n}$

$\therefore \displaystyle\sum_{1}^{\infty} b_n$ is cgt $\because$ I.G.S with $r < 1$

$\therefore \displaystyle\sum_{1}^{\infty} |a_n|$ is cgt by B.C.T ([illegible])

$\therefore \displaystyle\sum_{1}^{\infty} |a_n|$ is absolutely cgt.

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**Note** A series is said to be absolutely convergent if the series $\displaystyle\sum_{1}^{\infty} |a_n|$ is convergent

**DEFINITION** A Series $\displaystyle\sum_{1}^{\infty} a_n$ is said to be absolutely convergent if the Series $\displaystyle\sum_{1}^{\infty} |a_n|$ is convergent

**Q.21** $\sum_{1}^{\infty} \frac{\cos n\pi}{n+3} = \sum_{1}^{\infty} \frac{(-1)^n}{n+3}$

**Sol.** $a_n = \frac{(-1)^n}{n+3} \Rightarrow |a_n| = \frac{1}{n+3}$

Let $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{1}{n+3} \cdot \frac{n}{1}$

$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \frac{\cancel{n}}{\cancel{n}(1+\frac{3}{n})} = \boxed{1 \neq 0}$ ∴ both behave alike

∵ $\sum b_n = \sum \frac{1}{n}$ is dgt.

So by L comparison test $\sum a_n$ is divergent.

Al. Series Test

$|a_n| - |a_{n+1}| = \frac{1}{n+3} - \frac{1}{n+4} = \frac{n+4-n-3}{(n+3)(n+4)}$

$= \frac{1}{(n+3)(n+4)} > 0$ ∀ +ive values of $n$ i.e $n = 1, 2, 3, \dots$

2nd Method
$f(x) = \frac{1}{x+3}$
$f'(x) = \frac{-1}{(x+3)^2} < 0$
Seq is non increasing

$\Rightarrow |a_n| - |a_{n+1}| > 0$

$\Rightarrow |a_n| > |a_{n+1}|$ Hence non-increasing

ALTERNATING SERIES TEST
If $a_n > 0$ ∀ $n = 1, 2, 3, \dots$
Then A-Series is convergent if
(i) $a_n > a_{n+1}$
(ii) $\lim_{n\to\infty} a_n = 0$

Also $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n+3} = \boxed{0}$

Since both conditions for an alternating series test are satisfied so given Alternating Series is Convergent.

∴ Hence $\sum a_n$ is conditionally convergent.

**Q.22** $\sum_{1}^{\infty} \frac{n!}{(-2)^n}$

**Sol.** $\Rightarrow a_n = \frac{n!}{(-2)^n} \Rightarrow |a_n| = \frac{n!}{2^n}$

$|a_{n+1}| = \frac{(n+1)!}{2^{n+1}} = \frac{(n+1)n!}{2 \cdot 2^n}$

$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)n!}{2 \cdot 2^n} \cdot \frac{2^n}{n!} = \lim_{n\to\infty} \frac{n+1}{2} = \infty$

∴ Divergent

So by Ratio test Given Alternating Series test is dgt.

<u>Note</u> If Dgt Test, Ratio or Root test is applied to $\Sigma|a_n|$ and it is a <u>dgt series</u> then Al. Series is totally dgt. (No need of A.S.T.)

However if $\Sigma|a_n|$ dgs by B.C.T or L.C.T or Integral then we further apply Al.S.T for <u>cond cgt</u> or Tdgt.

Q23 B.2

## Q.23

$$\sum_0^\infty \frac{(-2)^n}{3^n+1}$$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$=\lim_{n\to\infty}\frac{2^{n+1}}{3^{n+1}+1}\cdot\frac{3^n+1}{2^n}$$

$$=\lim_{n\to\infty}\frac{\cancel{2^n}\cdot 2}{3^n\cdot 3+1}\cdot\frac{3^n+1}{\cancel{2^n}}$$

$$=\lim_{n\to\infty}\frac{2\cdot\cancel{3^n}(1+\frac{1}{3^n})}{\cancel{3^n}(3+\frac{1}{3^n})}$$

$$=\frac{2(1+0)}{3(3+0)}$$

$$=\frac{2}{3}<1$$

Hence Ab Cgt.

**Sol**

Here $a_n=\frac{(-2)^n}{3^n+1}$

$$|a_n| = \frac{2^n}{3^n+1}$$

$$|a_n|=\frac{2^n}{3^n+1} < \frac{2^n}{3^n} = |b_n| \text{ (say)}$$

$$\Sigma|b_n| = \Sigma\left(\frac{2}{3}\right)^n$$

$$=\left(\frac{2}{3}\right)^1+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\cdots$$

infinite G. Series with $r=\frac{2}{3}<1$

So $\Sigma\left(\frac{2}{3}\right)^n$ is convergent Series. Thus by comparison test given series is absolutely convergent.

## Q.25

$$\sum_1^\infty (-1)^{n+1}\frac{3n^2-3n+5}{n^3+n^2+n+1}$$

**Sol** $\Rightarrow$ $|a_n| = \dfrac{3n^2-3n+5}{n^3+n^2+n+1}$

<u>step 1</u> Let $\Sigma b_n=\sum_1^\infty \frac{1}{n}$

$$\frac{|a_n|}{b_n} = \lim_{n\to\infty}\frac{(3n^2-3n+5)}{n^3+n^2+n+1}\times\frac{n}{1}$$

$$=\lim_{n\to\infty}\frac{\cancel{n^3}(3-3/n+5/n^2)}{\cancel{n^3}(1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3})}$$

$$=\boxed{3}\neq 0$$

so both series behave alike. as $\Sigma b_n$ is dgt so $\Sigma a_n$ is dgt.

<u>step 2</u> using Alternating Series test

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{3n^2-3n+5}{n^3+n^2+n+1}$$

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{\cancel{n^2}(3-\frac{3}{n}+\frac{5}{n^2})}{n^{\cancel{3}}(1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3})}$$

$$\lim_{n\to\infty}|a_n| = \boxed{0}$$

Now Since $f(n) = \dfrac{3n^2-3n+5}{n^3+n^2+n+1}$

$$\Rightarrow f'(n) = \frac{(n^3+n^2+n+1)(6n-3)-(3n^2-3n+5)(3n^2+2n+1)}{(n^3+n^2+n+1)^2}$$

$$f'(n) = \frac{6n^4+6n^3+6n^2+6n-3n^3-3n^2-3n-3-9n^4+9n^3-15n^2-6n^3+6n^2 - 10n-3n^2+3n-5}{(n^3+n^2+n+1)^2}$$

$$f'(n) = \frac{-3n^4+6n^3-9n^2-4n-8}{(n^3+n^2+n+1)^2} \boxed{<0}$$ $\forall$ +ve values of $n$ i.e $n = 1,2,3 \ldots$

is decreasing sequence, so AST is cgt.

=) So given series is conditionally cgt.

**step 1** $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n}$

by stirling formula $\lfloor n = \sqrt{2\pi n}\, \frac{n^n}{e^n}$

$$\Rightarrow \lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{(3n^2-3n+5)}{n^3+n^2+n+1} \times \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n^2\left(3-\frac{3}{n}+\frac{5}{n^2}\right)n}{n^3\left(1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}\right)} = \lim_{n\to\infty} \frac{3-\frac{3}{n}+\frac{5}{n^2}}{1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}}$$

$$= 3 = l \neq 0$$

=) both series behaves alike. but $\sum_1^\infty b_n$ is divergent implies that the series Hence $\sum_1^\infty |a_n|$ is also divergent.

Given Series is conditionally ~~convergent~~

## Q.26 $\sum_1^\infty (-1)^n \frac{n^n}{n!}$

**SoL** Here $a_n = (-1)^n \frac{n^n}{n!} \Rightarrow |a_n| = \left|\frac{(-1)^n n^n}{n!}\right| = \frac{n^n}{n!}$

$$|a_{n+1}| = \frac{(n+1)^{n+1}}{(n+1)!}$$

then using ~~absolute~~ ratio test for absolute convergent i.e.

**Q.24** $\sum_{0}^{\infty}(-1)^n\left[\frac{\pi}{2}-\text{arc}\tan n\right]$

**Sol.** Here $a_n = (-1)^n\left(\frac{\pi}{2}-\tan^{-1}n\right)$

$\Rightarrow |a_n| = \frac{\pi}{2}-\tan^{-1}n$ so $f(x) = \frac{\pi}{2}-\tan^{-1}x$

$f'(x) = 0 - \frac{1}{1+x^2} < 0 \quad , x \geq 0$

$f(x)$ is non increasing on $[0,\infty[$

$\int_0^\infty f(x)dx$

$= \lim_{x\to\infty}\int_0^x\left(\frac{\pi}{2}-\tan^{-1}x\right)dx = \lim_{x\to\infty}\int_0^x \frac{\pi}{2}dx - \lim_{x\to\infty}\int_0^x \tan^{-1}x\,dx$

$$= \lim_{x\to\infty}\left|\frac{\pi}{2}x\right|_0^x - \lim_{x\to\infty}\int_0^x 1\cdot\tan^{-1}x\,dx$$

$$= \frac{\pi}{2}(\infty) - 0 - \lim_{x\to\infty}\left(\left|x\tan^{-1}x\right|_0^x - \frac{1}{2}\int_0^x\frac{2x}{1+x^2}dx\right)$$

$$= \infty - \lim_{x\to\infty}\left[x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)\right]_0^x$$

$$= \infty - \infty + \infty = \infty$$

$\Rightarrow \sum|a_n|$ is divergent

So we check it that alternating series is convergent or divergent

Thus $|a_n| - |a_{n+1}|$

$= \left(\frac{\pi}{2}-\tan^{-1}n\right) - \left(\frac{\pi}{2}-\tan^{-1}(n+1)\right)$

$= \tan^{-1}(n+1) - \tan^{-1}n > 0$

$\Rightarrow |a_n| - |a_{n+1}| > 0$ or $\boxed{|a_n| > |a_{n+1}|}$ $\{a_n\}$ non increasing seq.

**2nd Method**

$f(x) = \frac{\pi}{2}-\tan^{-1}x$

$f'(x) = -\frac{1}{1+x^2}\boxed{<0}$

$\therefore \{a_n\}$ is non-increasing

& $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\left|\frac{\pi}{2}-\tan^{-1}n\right| = \frac{\pi}{2}-\frac{\pi}{2} = \boxed{0}$

Since both conditions of an alternating series are satisfied but $\sum|a_n|$ is divergent Hence given series

So given alternating series is convergent

**Converges Conditionally**

**ALTERNATE** Here $|a_n| = \frac{\pi}{2}-\tan^{-1}n$
$= \cot^{-1}n$

put $\frac{\pi}{2}-\tan^{-1}n = \alpha$

$\Rightarrow \tan^{-1}n = \frac{\pi}{2}-\alpha$

$n = \tan\left(\frac{\pi}{2}-\alpha\right)$

$n = \cot\alpha$

$\Rightarrow \cot^{-1}n = \alpha$

so

$\frac{\pi}{2}-\tan^{-1}n = \cot^{-1}n$

$f(x) = \cot^{-1}x$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}$$

$$= \lim_{n\to\infty}\frac{(n+1)^n\cdot(n+1)}{(n+1)\, n!}\times\frac{n!}{n^n} = \lim_{n\to\infty}\frac{(n+1)^n}{n^n}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1$$

So by ratio test for absolute convergent, Given Series is divergent.

**Q.27** $\sum_{1}^{\infty}(-1)^{n+1}\frac{e^n}{n}$

**Sol** Here $a_n = (-1)^{n+1}\frac{e^n}{n} \Rightarrow |a_n| = \frac{e^n}{n}$

using root test for absolute convergent series

i.e $\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{e^n}{n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{e}{n^{\frac{1}{n}}}$ $\quad$ $\boxed{\because \lim_{n\to\infty} n^{\frac{1}{n}} = 1 \text{ already proved}}$

$$= \frac{e}{1} = e > 1$$

So by root test for absolute convergent Series, the given Series is divergent.

**Q.28** $\sum_{1}^{\infty}(-1)^{n+1}\frac{n^n}{3^{n^2}}$ $\quad$ $\because (3^{n^2})^{\frac{1}{n}} = (3^n)$

**Sol** Here $|a_n| = n^n/3^{n^2}$ using root test for absolute convergent series i.e $\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{n^n}{3^{n^2}}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{3^n}$ $\left(\frac{\infty}{\infty}\right)$

$$= \lim_{n\to\infty}\frac{1}{3^n \ln 3} \quad (\text{using L'Hospital Rule}) \quad \left(\because \frac{d}{dx}a^x = a^x \ln a\right)$$

$= \boxed{0 < 1}$ So by Root T. for A. Convergent given Series is absolute Convergent.

**Q.29** $\sum_{1}^{\infty}(-1)^n\frac{n^n}{a^{nb+c}}$ where $a > 1$, $b$ and $c$ are real

**Sol** Here $|a_n| = \frac{n^n}{a^{nb+c}}$ using Root Test for absolute convergent i.e

$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{n^n}{a^{nb+c}}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{a^{b+c/n}} = \frac{\infty}{a^b} = \infty$ So root test for absolute convergent, the given series is divergent.

using integral test

$$\int_0^\infty \cot^{-1}x\,dx = \lim_{t\to\infty}\int_0^t 1\cdot\cot^{-1}x\,dx$$

$$= \lim_{t\to\infty}\left[\left|x\cdot\cot^{-1}x\right|_0^t - \int_0^t \frac{-x}{1+x^2}\,dx\right]$$

$$= \lim_{t\to\infty}\left[t\cot^{-1}t - \cot^{-1}0 + \frac{1}{2}\left|\ln(1+x^2)\right|_0^t\right]$$

$$= \lim_{t\to\infty}\left[t\cot^{-1}t - \cot^{-1}0 + \frac{1}{2}\ln(1+t^2) - \frac{1}{2}\ln 2\right] = \infty$$

Hence $\sum_0^\infty |a_n|$ is divergent, by integral test.

Now as $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\cot^{-1}n = \cot^{-1}\infty = 0$

and $|a_n| - |a_{n+1}| > 0$ (already proved) $(\because \cot 0^\circ = \infty)$

Hence given series is convergent by A. S. test. but $\sum_0^\infty |a_n|$ is divergent. So given series is conditionally convergent.

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Q. 30—35. find value of $x$ for which the given series

(i) absolutely convergent (ii) Conditionally convergent

(iii) Divergent.

**Q. 30** $\sum_1^\infty \frac{nx^n}{3^n}$

**SOL.** Here $|a_n| = \left|\frac{nx^n}{3^n}\right|$

$$|a_{n+1}| = \frac{|n+1||x^{n+1}|}{|3^{n+1}|}$$

Root Test 2nd Method

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left|\frac{nx^n}{3^n}\right|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\frac{n^{\frac{1}{n}}|x|}{3}$$

$$= 1\cdot\frac{|x|}{3}$$

$$= \frac{|x|}{3}$$

using Ratio Test for absolutely convergent

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{|n+1|\,|x^n\cdot x|}{|3\cdot 3^n|}\times\frac{|3^n|}{|n||x^n|} = \lim_{n\to\infty}\frac{|n+1|\,|x|}{|3|\cdot|n|}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)\left|\frac{x}{3}\right| = \lim_{n\to\infty}\frac{n}{n}\left(1+\frac{1}{n}\right)\frac{|x|}{3} = \frac{|x|}{3}$$

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Thus given series by ratio test absolutely converges for $\frac{|x|}{3} < 1$ i.e for $|x| < 3$

— diverges for $\frac{|x|}{3} > 1$ i.e for $|x| > 3$

Also if $\frac{|x|}{3} = 1 \Rightarrow |x| = 3 \Rightarrow x = \pm 3$

then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n(\pm 3)^n}{3^n} = \mp\infty \neq 0$

For $x = 3$: $\lim_{n\to\infty} \frac{n(3^n)}{(3^n)} = \infty$

For $x = -3$: $\lim_{n\to\infty} \frac{n(-3)^n}{(3^n)} = \lim_{n\to\infty} \frac{n(-1)^n 3^n}{(3^n)} = \infty$

so by divergent test the given series diverges.

for $x = \pm 3$ ⟶ (ii)

Combining statement ① and ②, we concluded that the given series is divergent for $|x| \geq 3$

## Q.31 $\sum_1^\infty \frac{x^n}{\sqrt{n}}$

Sol Here $|a_n| = \left|\frac{x^n}{\sqrt{n}}\right| \Rightarrow |a_n| = \frac{|x^n|}{|\sqrt{n}|} = \frac{|x^n|}{\sqrt{n}}$

and $|a_{n+1}| = \frac{|x^{n+1}|}{\sqrt{n+1}} = \frac{|x||x^n|}{\sqrt{n+1}}$

2nd Method $\boxed{\text{By Root test}}$

$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{|x|}{\sqrt{n^{\frac{1}{n}}}} = |x|$.

using Ratio test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|\cancel{|x^n|}}{\sqrt{n+1}} \times \frac{\sqrt{n}}{\cancel{|x^n|}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} |x|$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{n+1}{n}}} |x| = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} |x|$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \longrightarrow ①$$

if $|x| < 1$ Series is Ab. Cgt

if $|x| > 1$ divergent Series

if $|x| = 1$ Test Fail

$|x| = 1 \Rightarrow x = \pm 1$, For $x = 1$ $\sum_1^\infty \frac{1^n}{\sqrt{n}} = \sum_1^\infty \frac{1}{\sqrt{n}} \Rightarrow a_n = \frac{1}{\sqrt{n}}$

Which is divergent. Thus the Series is also divergent for $x = 1$. and

For $x=-1$ $\sum_{1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ which is an alternating series. So using alternating series test

$f(x)=\frac{1}{\sqrt{x}}$
$f'(x)=-\frac{1}{2}x^{-3/2}$
$f'(x)<0$
$\therefore$ non increasing

i.e $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\left|\frac{1}{\sqrt{n}}\right| = 0 \quad \left(\because a_n=\frac{(-1)^n}{\sqrt{n}}\right)$

and $|a_n| = \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} = |a_{n+1}|$ for all $n\geq 1$.

So by alternating series test, series $\sum_{1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges but $\sum_{1}^{\infty}\frac{1}{\sqrt{n}} = \sum_{1}^{\infty}|a_n|$ is divergent, so by the definition of conditionally convergent, the series conditionally converges for $x=-1$.

x ———————— x

We can use Ratio also

Root Test

$\sum a_n = \sum \frac{4^n}{x^n}$

$|a_n|^{\frac{1}{n}} = \left|\frac{4}{x}\right|^{n\cdot\frac{1}{n}}$

$= \left|\frac{4}{x}\right|$

Series is cgt if $\left|\frac{4}{x}\right|<1$

dgt if $\left|\frac{4}{x}\right|>1$

Fails for $\left|\frac{4}{x}\right|=1$

i.e $|x|=4$

$\Rightarrow x=\pm 4$

For $x=4$

For $x=-4$

**Q. 32** $\sum_{0}^{\infty}\frac{4^n}{x^n}$

**SOL.** Here $a_n = \frac{4^n}{x^n} \Rightarrow |a_n| = \left|\frac{4^n}{x^n}\right| = \frac{4^n}{|x^n|}$

$= \left|\frac{4}{x}\right|^n = \left(\frac{4}{|x|}\right)^n$ Easy

I.G.S cgs if $r<1$

$\therefore$ I.G.S converges for $r=\frac{4}{x}<1 \Rightarrow |x|>4$

I.G.S Diverges for $\frac{4}{|x|}>1 \Rightarrow |x|<4$

I.G.S Diverges for $\frac{4}{|x|}=1 \Longrightarrow |x|=4 \Rightarrow x=\pm 4$

Also when $x=0$, the series is not defined. Therefor it is divergent for $0<|x|\leq 4$

x ———————— x

**Q. 33** $\sum_{0}^{\infty}\frac{(-1)^n x^n}{3^n(n+2)!}$

Sol $a_n = \frac{(-1)^n x^n}{3^n(n+2)!} \Rightarrow |a_n| = \left|\frac{(-1)^n x^n}{3^n(n+2)!}\right| = \frac{|x|^n}{3^n(n+2)!}$

So $|a_{n+1}| = \frac{|x|^{n+1}}{3^{n+1}(n+3)!}$

using ratio test for absolute convergent i.e

$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{|x||x|^n}{3.3^n(n+3)!}\times\frac{3^n(n+2)!}{|x|^n} = \lim_{n\to\infty}\frac{|x|(n+2)!}{3(n+3)(n+2)!}$

$$= \lim_{n\to\infty} \frac{|x|}{3(n+3)} = 0 < 1$$

So by ratio-test for absolute convergence, the given alternating series is absolutely convergent for all values of $x$.

x ———— x

## Q.34 $\sum_{1}^{\infty} (-1)^{n-1} \frac{x^n}{n(n+1)}$

**SOL** $\Rightarrow |a_n| = \left|\frac{x^n}{n(n+1)}\right| = \frac{|x^n|}{n(n+1)} = \frac{|x|^n}{n(n+1)}$

and $|a_{n+1}| = \frac{|x|^{n+1}}{(n+1)(n+2)}$

using ratio test for absolute convergent i.e

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{(\cancel{n+1})(n+2)} \times \frac{n(\cancel{n+1})}{|x|^n} = \lim_{n\to\infty} \frac{n|x|}{n+2}$$

$$= \lim_{n\to\infty} \frac{|x|}{\frac{n+2}{n}} = \lim_{n\to\infty} \frac{|x|}{1+\frac{2}{n}} = \frac{|x|}{1+0} = |x|$$

The series converges absolutely for $|x|<1$. and diverges for $|x|>1$.

Also if $|x| = 1$ i.e $x = 1, -1$.

when $x=1$, then given series will be $\sum_{1}^{\infty} (-1)^{n-1} \frac{1}{n(n+1)}$

for $x=-1$, " " $\sum_{1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n(n+1)} = \sum_{1}^{\infty} (-1)^{2n-1} \frac{1}{n(n+1)}$

In both cases

$\frac{1}{n(n+1)} = \frac{1}{n^2+n} < \frac{1}{n^2} = b_n$ which is cgt so for $x=\pm1$ given series is Absolutely Cgt

Thus the series is absolutely convergent for $|x| \le 1$ and divergent for $|x|>1$.

x ———— x

## Q.35 $\sum_{0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

**SOL** Here $|a_n| = \left|\frac{x^{2n}}{2^{2n}(n!)^2}\right| = \frac{|x|^{2n}}{2^{2n}(n!)^2}$

$\Rightarrow |a_{n+1}| = \frac{|x|^{2n+2}}{2^{2n+2}[(n+1)!]^2}$

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sing ratio test for absolute convergent series ...

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{2n+2}}{2^{2n+2}\left[(n+1)!\right]^2} \times \frac{2^{2n}(n!)^2}{|x|^{2n}}$$

$$= \lim_{n\to\infty} \frac{|x|^2 \cdot 2^{2n}(n!)^2}{2^2\, 2^{2n}\,(n+1)^2 (n!)^2}$$

$$= \lim_{n\to\infty} \frac{x^2}{4(n+1)^2} = 0 < 1$$

$\Rightarrow$ the given series is absolutely convergent for all values of $x$.