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Notes: Chapter 5

The Definite Integral

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⇒ Definite integral as Limit of a Sum

Definition:-

Let f be a continuous real-valued function defined on a finite closed interval $[a, b]$. A partition P of $[a, b]$ is a finite set of points.

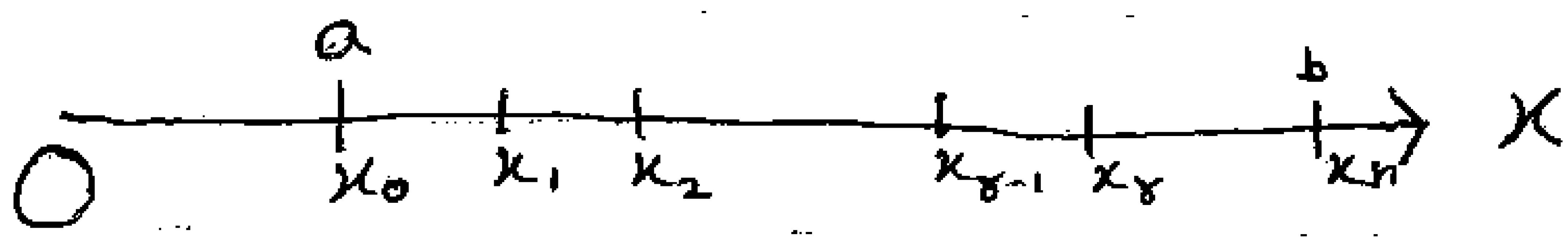
$$P = \{x_0, x_1, x_2, x_3, \text{-----}, x_{n-1}, x_n\}$$

So,

$$a = x_0 < x_1 < x_2 < x_3 \text{-----} < x_{n-1} < x_n = b$$

It subdivides into n closed subinterval.

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \text{------}, [x_{n-1}, x_n].$$



The rth Subinterval $[x_{r-1}, x_r]$ and its length $x_r - x_{r-1}$ Will both be denoted by Δx_r.

The norm of P as such:-

$$\|P\| = \max_{1 \le r \le n} \Delta x_r$$

Let c_r be any point of $[x_{r-1}, x_r]$; $r = 1, 2, 3, \ldots, n$

The Expression,

$$(x_1 - x_0) f(c_1) + (x_2 - x_1) f(c_2) + \cdots + (x_r - x_{r-1}) f(c_r) + \cdots$$

$$\cdots + (x_n - x_{n-1}) f(c_n)$$

$$= \sum_{r=1}^{n} (x_r - x_{r-1}) f(c_r)$$

$$= \sum_{r=1}^{n} \Delta x_r \, f(c_r)$$

is called Reimann Sum. This Sum denoted by $S(P, \delta)$.

$$\int_a^b f(x)\, dx \quad \text{or} \quad \int_a^b f$$

In this case, f is said to be integrable over $[a, b]$

The numbers a and b are called Lower and upper Limits of integration.

CH No. 5

Evaluation of definite integrals

⇒ **Arithmetic Sub-division formula :-**

Let $f(x)$ be defined on $[a,b]$

$$\Delta x = \frac{b-a}{n} = \text{length of Sub-Interval.}$$

Partition P of $[a,b]$.

$$P = \{a, a+\Delta x, a+2\Delta x, a+3\Delta x, \ldots, a+(n-1)\Delta x, a+n\Delta x\}$$

So, Sub-Intervals :-

$$[a, a+\Delta x], [a+\Delta x, a+2\Delta x], [a+2\Delta x, a+3\Delta x], \ldots, [a+(n-1)\Delta x, a+n\Delta x].$$

By Reimenn Sum :- Taking c_r as left end or right end of each Sub-Intervals.

$$S(P,f) = (a+\Delta x - a) f(a) + (a+2\Delta x-(a+\Delta x)) f(a+\Delta x) + (a+3\Delta x-(a+2\Delta x)) f(a+2\Delta x) + \ldots + (a+n\Delta x-(a+(n-1)\Delta x)) f(a+(n-1)\Delta x)$$

$$= \Delta x f(a) + \Delta x f(a+\Delta x) + \Delta x f(a+2\Delta x) + \ldots + \Delta x f(a+(n-1)\Delta x).$$

$$= \Delta x\left[f(a) + f(a+\Delta x) + f(a+2\Delta x) + \ldots + f(a+(n-1)\Delta x)\right]$$

Taking Lim as $n \to \infty$ then, $\Delta x \to 0$

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \Delta x \to 0}} \Delta x\left[f(a) + f(a+\Delta x) + f(a+2\Delta x) + \ldots + f(a+(n-1)\Delta x)\right]$$

And, Now using formula and Solve the Question of Exercise.

⇒ **Geometric Sub-division formula :-**

Let $f(x)$ is defined on $[a,b]$.

Partition $P = \{a, ar, ar^2, ar^3, ar^4, \ldots, ar^{n-1}, ar^n\}$

Sub-Intervals :-

$$[a, ar], [ar, ar^2], [ar^2, ar^3], \ldots [ar^{n-1}, ar^n].$$

By Reimenn Sum, Taking c_r as left end point of each Sub-Intervals

$S(P,f) = (ar-a)f(a) + (ar^2-ar)f(ar) + (ar^3-ar^2)f(ar^2) + \cdots\cdots +$
$(ar^n - ar^{n-1})f(ar^{n-1})$

$= a(r-1)f(a) + ar(r-1)f(ar) + (ar^2)(r-1)f(ar^2) + \cdots\cdots +$
$ar^{n-1}(r-1)f(ar^{n-1}).$

$= (r-1)\left[af(a) + arf(ar) + ar^2f(ar^2) + \cdots\cdots + ar^{n-1}f(ar^{n-1})\right].$

Taking Lim $n \to \infty$ as $r \to 1$ and $r^n = \frac{b}{a}$

$\int_a^b f(x)\,dx = \lim_{r\to 1}\left[(r-1)\left\{af(a) + arf(ar) + ar^2f(ar^2) + \cdots\cdots + ar^{n-1}f(ar^{n-1})\right\}\right].$

And using this formula:-

$S_n = \frac{a_1(r^n-1)}{r-1}$ or $S_n = \frac{a_1(1-r^n)}{(1-r)}$

Ex # 5.1

Qno.1:- $\int_{-1}^{1} x\,dx$

Sol:- $\int_a^b f(x)\,dx$; $f(x) = x \Rightarrow f(a) = a$

BY Geometric Subdivision formula:-

Let $P = \{a, ar, ar^2, ar^3, \cdots\cdots, ar^{n-1}, ar^n\}$ be Partition of $[a,b]$.

Sub-Intervals:- $[a, ar], [ar, ar^2], [ar^2, ar^3], \cdots\cdots, [ar^{n-1}, ar^n]$

BY Reimenn Sum Taking c_r as left end point of each Sub-interval

$S(P,f) = (ar-a)f(a) + (ar^2-ar)f(ar) + (ar^3-ar^2)f(ar^2) + \cdots\cdots + (ar^n - ar^{n-1})$
$f(ar^{n-1}).$

$= a(r-1)f(a) + ar(r-1)f(ar) + ar^2(r-1)f(ar^2) + \cdots\cdots + ar^{n-1}(r-1)$
$f(ar^{n-1}).$

$= (r-1)\left[af(a) + arf(ar) + ar^2f(ar^2) + \cdots\cdots + ar^{n-1}f(ar^{n-1})\right].$

$= (r-1)\left[a \cdot a + ar \cdot ar + ar^2 \cdot ar^2 + \cdots\cdots + ar^{n-1} \cdot ar^{n-1}\right]$

$$S(P,f) = (r-1)\{a^2 + a^2r^2 + a^2r^4 + \cdots + a^2 r^{(n-1)2}\}$$

$$= (r-1)\,a^2\,[\,1 + r^2 + r^4 + \cdots + \text{to } n\text{-terms}]$$

$\Rightarrow (r-1)\,a^2$ and there $a_1 = 1$, $r = r^2$, $n = n$

So, $S_n = \dfrac{1((r^2)^n - 1)}{(r^2-1)} = \dfrac{r^{2n}-1}{(r^2-1)}$

And,

$$S(P,f) = (r-1)\,a^2 \cdot \frac{(r^n)^2 - 1}{(r^2-1)} = (\cancel{r-1})\,a^2 \frac{(r^n)^2 - 1}{(r+1)(\cancel{r-1})}$$

$$\Rightarrow r^n = \frac{b}{a}$$

$$= a^2\,\frac{(b/a)^2 - 1}{(r+1)} = \frac{\cancel{a^2} \cdot b^2 - a^2 / \cancel{a^2}}{(r+1)}$$

Taking $\lim n \to \infty$ as $r \to 1$:-

$$\int_a^b x\,dx = \lim_{r\to 1} \frac{b^2 - a^2}{(r+1)}$$

Also, $\displaystyle\int_{-1}^{1} x\,dx = \lim_{r\to 1} \frac{(1)^2 - (-1)^2}{r+1} = \frac{1-1}{1+1} = \frac{0}{2}$

$$\int_{-1}^{1} x\,dx = 0 \quad \text{Ans}$$

Q No. 2:- $\displaystyle\int_a^b \frac{1}{x}\,dx$

$\displaystyle\int_a^b f(x)\,dx \quad \Rightarrow \quad f(x) = \frac{1}{x}$

BY Geometric Sub-division formula:-

Partition will be:-

$P = \{a,\ ar,\ ar^2,\ ar^3,\ ar^4, \ldots, ar^{n-1},\ ar^n\}$

Sub-intervals:-

$[a, ar], [ar, ar^2], [ar^2, ar^3], \ldots, [ar^{n-1}, ar^n]$.

~~By Reimann Sum Taking r as left end point of Sub-Interval~~

$S(P,f) = (ar-a)f(a) + (ar^2-ar)f(ar) + (ar^3-ar^2)f(ar^2) + \cdots +$

$(ar^n - ar^{n-1})f(ar^{n-1})$

$= (r-1)\left[af(a) + arf(ar) + ar^2 f(ar^2) + \cdots + ar^{n-1}f(ar^{n-1})\right]$

$= (r-1)\left[a.\frac{1}{a} + ar\frac{1}{ar} + ar^2\frac{1}{ar^2} + \cdots + ar^{n-1}\frac{1}{ar^{n-1}}\right]$

$= (r-1)\left[1+1+1+1+ \cdots + n\text{-terms}\right]$

$= (r-1)\left[n(1)\right]$

$= (r-1)\,n$

We know that $r^n = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{1/n}$

$\ln r = \frac{1}{n}\ln\left(\frac{b}{a}\right)$

$n = \frac{1}{\ln r}\ln\left(\frac{b}{a}\right)$;

Taking $\lim\limits_{n\to\infty}$ as $r \to 1$.

$\lim\limits_{n\to\infty} S(P,f) = \lim\limits_{r\to 1}\left[(r-1).\frac{1}{\ln r}\ln\left(\frac{b}{a}\right)\right]$

$= \ln\left(\frac{b}{a}\right).\lim\limits_{r\to 1}\left(\frac{r-1}{\ln r}\right)$

By L-Houpital rule.

$= \ln\left(\frac{b}{a}\right).1$

$\int_a^b \frac{1}{x}\,dx = \ln b - \ln a$ Ans

Q No. 3:- $\int_a^b x^2 dx$

$\int_a^b f(x)\,dx \quad \Rightarrow f(x) = x^2$

By Geometric sub-division formula:-

Let $P = \{a, ar, ar^2, ar^3, \ldots, ar^{n-1}, ar^n\}$ be a Partition of $[a, b]$

Sub-Intervals:-

$[a, ar], [ar, ar^2], [ar^2, ar^3], \ldots, [ar^{n-1}, ar^n]$.

By Reimenn Sum, Taking c_r as left end point of each Sub-Interval.

$$S(P,f) = (ar - a)f(a) + (ar^2 - ar)f(ar) + (ar^3 - ar^2)f(ar^2) + \cdots + (ar^n - ar^{n-1})f(ar^{n-1})$$

$$= (r-1)\{a\,f(a) + ar\,f(ar) + ar^2 f(ar^2) + \cdots + ar^{n-1} f(ar^{n-1})\}$$

$$= (r-1)[a \cdot a^2 + ar \cdot (ar)^2 + ar^2 (ar^2)^2 + \cdots + ar^{n-1} \cdot (ar^{n-1})^2]$$

$$S(P,f) = (r-1)[a^3 + a^3 r^3 + a^3 r^6 + \cdots + \text{to } n\text{-terms}]$$

$$= (r-1)a^3[1 + r^3 + r^6 + \cdots + \text{to } n\text{-terms}]$$

So, $a_1 = 1$, $r = r^3$, $n = n$.

$$S_n = \frac{1((r^3)^n - 1)}{(r^3 - 1)} = \frac{r^{3n} - 1}{(r^3 - 1)} = \frac{(r^n)^3 - 1}{(r^3 - 1)}$$

$$S(P,f) = (\cancel{r-1})\, a^3 \cdot \frac{(r^n)^3 - 1}{(\cancel{r-1})(r^2 + r + 1)} = a^3 \cdot \frac{(b/a)^3 - 1)}{(r^2 + r + 1)} = a^3 \frac{(b^3 - a^3)/a^3}{(r^2 + r + 1)}$$

Taking lim $n \to \infty$ as $r \to 1$.

$$\int_a^b x^2 dx = \lim_{r \to 1} \frac{a^3 (b^3 - a^3)/a^3}{r^2 + r + 1} = \lim_{r \to 1} \frac{\cancel{a^3} (b^3 - a^3)/\cancel{a^3}}{r^2 + r + 1}$$

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{(1)^2 + 1 + 1} = \frac{b^3 - a^3}{3} \quad \text{Ans}$$

Q No. 4 :- $\int_a^b \frac{1}{\sqrt{x}}\,dx$

$\int_a^b f(x)\,dx \Rightarrow f(x) = \frac{1}{\sqrt{x}}$

By Geometric Sub-division formula:-

Let $P = \{a, ar, ar^2, ar^3, ----, ar^{n-1}, ar^n\}$ be Partition of $[a,b]$.

Sub-Intervals:-

$[a, ar], [ar, ar^2], [ar^2, ar^3], -----, [ar^{n-1}, ar^n]$.

By Reimenn Sum Taking c_r as left end point of each Intervals.

$S(P,f) = (ar-a)f(a) + (ar^2-ar)f(ar) + (ar^3-ar^2)f(ar^2) + ------ + (ar^n - ar^{n-1})f(ar^{n-1})$.

$= (r-1)[af(a) + ar\,f(ar) + ar^2 f(ar^2) + ----- + (ar^{n-1})f(ar^{n-1})]$

$= (r-1)\left[a.\frac{1}{\sqrt{a}} + ar.\frac{1}{\sqrt{ar}} + ar^2.\frac{1}{\sqrt{ar^2}} + ------ + ar^{n-1}.\frac{1}{\sqrt{ar^{n-1}}}\right]$

$S(P,f) = (r-1)[\sqrt{a} + \sqrt{ar} + \sqrt{ar^2} + ----- + \text{to } n\text{-terms}]$

$= (r-1)\sqrt{a}\{1 + \sqrt{r} + \sqrt{r^2} + ------ \text{to } n\text{-terms}\}$

So, where, $a = 1$, $r = \sqrt{r}$, $n = n$

$S_n = \frac{1((\sqrt{r})^n - 1)}{(\sqrt{r}-1)}$

$S(P,f) = (r-1)\sqrt{a}\,\frac{\sqrt{r^n}-1}{\sqrt{r}-1} = \sqrt{a}(\sqrt{r}+1)\cancel{(\sqrt{r}-1)}\frac{(\sqrt{r^n}-1)}{\cancel{(\sqrt{r}-1)}}$

$S(P,f) = \sqrt{a}(\sqrt{r}+1)\left(\sqrt{\frac{b}{a}} - 1\right)$

Taking $\lim n \to \infty$ as $r \to 1$:-

$\lim_{n\to\infty} S(P,f) = \lim_{\substack{n\to\infty \\ r\to 1}} \sqrt{a}(\sqrt{r}+1)\frac{(\sqrt{b}-\sqrt{a})}{\sqrt{a}}$

$\int_a^b \frac{1}{\sqrt{x}}\,dx = 2[\sqrt{b} - \sqrt{a}]$ Ans

Q No 5:- $\int_a^b \sin x\, dx$

$\int_a^b f(x)\, dx \Rightarrow f(x) = \sin x$

Length of each Sub-Interval $= \Delta x = \frac{b-a}{n}$

By arithmetic Sub-division formula:-

Partition $= P = \{a, a+\Delta x, a+2\Delta x, a+3\Delta x, \ldots, a+n\Delta x\}$

Sub-Intervals:-

$[a, a+\Delta x], [a+\Delta x, a+2\Delta x], [a+2\Delta x, a+3\Delta x], \ldots [a+(n-1)\Delta x, a+n\Delta x]$.

By Reimenn Sum, c_x Taking as left end Point.

$$S(P,f) = (a+\Delta x - a) f(a) + (a+2\Delta x - (a+\Delta x)) f(a+\Delta x) + \ldots + (a+n\Delta x - (a+(n-1)\Delta x)) f(a+(n-1)\Delta x)$$

$$= \Delta x\, f(a) + \Delta x\, f(a+\Delta x) + \Delta x\, f(a+2\Delta x) + \ldots + \Delta x\, f(a+(n-1)\Delta x)$$

$$= \Delta x\left[f(a) + f(a+\Delta x) + f(a+2\Delta x) + \ldots + \text{to } n\text{-terms}\right]$$

$$= \Delta x\left[\sin a + \sin(a+\Delta x) + \sin(a+2\Delta x) + \ldots + \text{to } n\text{-terms}\right]$$

We know that;

$$S(P,f) = \Delta x \cdot \frac{\sin\left(a + \frac{n-1}{2}\Delta x\right) \sin\frac{n\Delta x}{2}}{\sin\frac{\Delta x}{2}}$$

$$= \Delta x \cdot \frac{\sin\left(a + \left(\frac{b-a}{\Delta x} - 1\right)/2 \cdot \Delta x\right) \sin\left(\frac{b-a}{\Delta x} \cdot \frac{\Delta x}{2}\right)}{\sin\frac{\Delta x}{2}}$$

$$= \Delta x \cdot \frac{\sin(2a+b-a-\Delta x)/2\, \sin(b-a)/2}{\sin \Delta x/2}$$

$$= \Delta x \frac{\sin(a+b-\Delta x)/2\, \sin(b-a)/2}{\sin\frac{\Delta x}{2}}$$

$$= \frac{2\sin(a+b-\Delta x)/2\, \sin(b-a)/2}{\frac{\sin \Delta x/2}{\Delta x/2}}$$

Taking $\lim_{n\to\infty}$ then, $\Delta x \to 0$

$$\int_a^b \sin x\, dx = \lim_{\Delta x\to 0} \frac{2\sin(a+b-\Delta x)/2\, \sin(b-a)/2}{\sin \Delta x/2 \,/\, \Delta x/2}$$

$$\int_a^b \sin x\, dx = \frac{2\sin(a+b)/2 \sin(b-a)/2}{1}$$

$$\int_a^b \sin x\, dx = \cos\left(\frac{a+b-b+a}{2}\right) - \cos\left(\frac{a+b+b-a}{2}\right)$$

$$= \cos\left(\frac{2a}{2}\right) - \cos\left(\frac{2b}{2}\right)$$

$$\int_a^b \sin x\, dx = \cos a - \cos b \quad \text{Ans}$$

Q NO. 6:- $\int_a^b \sin^2 x\, dx$

$\int_a^b f(x)\, dx \Rightarrow f(x) = \sin^2 x$

Length of each sub-Interval $= \Delta x = \frac{b-a}{n}$

Partition of $[a,b]$ is:-

$P = \{a, a+\Delta x, a+2\Delta x, a+3\Delta x, \ldots\ldots, a+(n-1)\Delta x, a+n\Delta x\}$

Sub-Intervals:-

$[a, a+\Delta x], [a+\Delta x, a+2\Delta x], [a+2\Delta x, a+3\Delta x], \ldots\ldots, [a+(n-1)\Delta x, a+n\Delta x]$

By Reimenn Sum Taking c_r as left end point of each S·Interval.

$$(P,f) = \Delta x f(a) + \Delta x f(a+\Delta x) + \Delta x f(a+2\Delta x) + \ldots\ldots + \Delta x f(a+(n-1)\Delta x)$$

$$= \Delta x \{ f(a) + f(a+\Delta x) + f(a+2\Delta x) + \ldots\ldots + \text{to } n \text{ terms} \}$$

$$= \Delta x \{ \sin^2 a + \sin^2(a+\Delta x) + \sin^2(a+2\Delta x) + \ldots\ldots + \text{to } n\text{-terms} \}$$

We know that, $1 - \cos 2x = 2\sin^2 x$

$$\Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$$

So,

$$(P,f) = \Delta x \left\{ \frac{1-\cos 2a}{2} + \frac{1-\cos 2(a+\Delta x)}{2} + \frac{1-\cos 2(a+2\Delta x)}{2} + \ldots\ldots + \text{to } n\text{-terms} \right\}$$

$$= \Delta x \left[\left\{ \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots + \text{to } n\text{-terms} \right\} - \frac{1}{2} \left\{ \cos 2a + \cos(2a+2\Delta x) + \cos(2a+4\Delta x) + \ldots\ldots + \text{to } n\text{-terms} \right\} \right]$$

$$S(P,f) = \Delta x\left[n\left(\tfrac{1}{2}\right) - \tfrac{1}{2}\left\{ \cos 2a + \cos(2a+2\Delta x) + \cos(2a+4\Delta x) + \cdots + \text{to } n\text{-terms} \right\}\right]$$

$$= \Delta x\left[\frac{1}{2}\cdot\frac{(b-a)}{\Delta x} - \frac{1}{2}\left\{ \frac{\cos\left(2a + \frac{(n-1)}{2}\cdot 2\Delta x\right)\cdot \sin\frac{n(2\Delta x)}{2}}{\left(\sin\frac{2\Delta x}{2}\right)} \right\}\right]$$

$$= \cancel{\Delta x}\,\frac{1}{2}\cdot\frac{b-a}{\cancel{\Delta x}} - \frac{1}{2}\,\Delta x\left\{ \frac{\cos\left(2a + \left(\frac{b-a}{\Delta x} - 1\right)\Delta x\right)\cdot \sin\left(\frac{b-a}{\Delta x}\right)\cdot(\Delta x)}{\sin \Delta x} \right\}$$

$$= \frac{b-a}{2} - \frac{1}{2}\,\Delta x\left\{ \frac{\cos(2a+b-a-\Delta x)\cdot \sin(b-a)}{\sin \Delta x} \right\}$$

$$= \frac{b-a}{2} - \frac{1}{2}\,\Delta x\left\{ \frac{\cos(a+b-\Delta x)\cdot \sin(b-a)}{\sin \Delta x} \right\}$$

Taking $\lim_{n\to\infty}$ as $\Delta x \to 0$

$$\int_a^b \sin^2 x\,dx = \lim_{\Delta x\to 0}\left[\frac{b-a}{2} - \frac{1}{2}\left\{ \frac{\cos(a+b-\Delta x)\cdot\sin(b-a)}{\frac{\sin\Delta x}{\Delta x}} \right\}\right]$$

$$= \frac{b-a}{2} - \frac{1}{2}\,\frac{\lim_{\Delta x\to 0}\cos(a+b-\Delta x)\cdot\sin(b-a)}{\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}}$$

$$= \frac{b-a}{2} - \frac{1}{2}\cos(a+b)\cdot\sin(b-a)$$

$$= \frac{b-a}{2} - \frac{1}{2}\cdot\frac{2}{2}\cos(a+b)\cdot\sin(b-a)$$

$$= \frac{b-a}{2} - \frac{1}{4}\left[\sin(\cancel{a}+b+b-\cancel{a}) - \sin(a+\cancel{b}-\cancel{b}+a)\right]$$

$$= \frac{b-a}{2} - \frac{1}{4}(\sin 2b - \sin 2a) \text{ Ans}$$

Q No. 7:- $\int_a^b \cosh x\, dx$

$\int_a^b f(x)\,dx \Rightarrow f(x) = \cosh x$

Length of each Sub-Interval $= \Delta x = \frac{b-a}{n}$

BY Arithmetic Sub-division :-

Partition of $[a,b] = P = [a, a+\Delta x, a+2\Delta x, a+3\Delta x, \ldots\ldots, a+n\Delta x]$

Sub-Intervals :-

$[a, a+\Delta x], [a+\Delta x, a+2\Delta x], [a+2\Delta x, a+3\Delta x], \ldots\ldots, [a+(n-1)\Delta x, a+n\Delta x]$.

BY Reimenn Sum, Taking c_r as left end point.

$$S(P,f) = \Delta x\, f(a) + \Delta x\, f(a+\Delta x) + \Delta x (f(a+2\Delta x) + \ldots\ldots + \text{to } n\text{-terms}$$

$$= \Delta x \{ f(a) + f(a+\Delta x) + f(a+2\Delta x) + \ldots\ldots + \text{to } n\text{-terms} \}$$

$$= \Delta x \{ \cosh a + \cosh(a+\Delta x) + \cosh(a+2\Delta x) + \ldots\ldots + \text{to } n\text{-terms} \}$$

we know that, $\cosh x = \frac{e^x + e^{-x}}{2}$

So,

$$S(P,f) = \Delta x \left\{ \frac{e^a + e^{-a}}{2} + \frac{e^{a+\Delta x} + e^{-(a+\Delta x)}}{2} + \frac{e^{a+2\Delta x} + e^{-(a+2\Delta x)}}{2} + \ldots\ldots + \text{to } n\text{-terms} \right\}$$

$$= \frac{\Delta x}{2} \left\{ (e^a + e^{a+\Delta x} + e^{a+2\Delta x} + \ldots\ldots + \text{to } n\text{-terms}) + (e^{-a} + e^{-a-\Delta x} + e^{-a-2\Delta x} + \ldots\ldots \text{ to } n\text{- terms}) \right\}$$

$$= \frac{\Delta x}{2} \left\{ e^a (1 + e^{\Delta x} + e^{2\Delta x} + \ldots\ldots + \text{to } n\text{-terms}) + e^{-a}(1 + e^{-\Delta x} + e^{-2\Delta x} + \ldots\ldots + \text{to } n\text{- terms}) \right\}$$

$$= \frac{\Delta x}{2} \{ \text{where } a_1 = 1, r = e^{\Delta x}, n = n \; + \; a_1 = 1, r = e^{-\Delta x}, n = n \}$$

$$= \frac{\Delta x}{2} \left\{ e^a \left(\frac{1-(e^{\Delta x})^n}{1-e^{\Delta x}} \right) + e^{-a} \left(\frac{1-(e^{-\Delta x})^n}{1-e^{-\Delta x}} \right) \right\}$$

$$= \frac{\Delta x\, e^a (1-(e^n)^{\Delta x})}{2\,(1-e^{\Delta x})} + \frac{e^{-a} \Delta x (1-(e^n)^{-\Delta x})}{2\,(1-e^{-\Delta x})}$$

Taking $\lim_{n\to\infty}$ as $\Delta x \to 0$.

$$\lim_{n\to\infty} S(P,f) = \lim_{\Delta x\to 0} \frac{e^a \Delta x \left(1 - e^{(\frac{b-a}{\Delta x})\cdot \Delta x}\right)}{2\,(1-e^{\Delta x})} + \lim_{\Delta x \to 0} \frac{e^{-a} \Delta x \left(1 - e^{(\frac{b-a}{\Delta x})\cdot -\Delta x}\right)}{2\,(1-e^{-\Delta x})}$$

$$= \frac{e^a (1-e^{(b-a)})}{2 \lim\limits_{\Delta x\to 0} \left(\frac{e^{\Delta x}-1}{\Delta x} \right)} + \frac{e^{-a} (1-e^{-(b-a)})}{2 \lim\limits_{\Delta x\to 0} \left(\frac{e^{-\Delta x}-1}{-\Delta x} \right)}$$

$$\int_a^b \cosh x\,dx = \frac{e^a}{2}\cdot\frac{1-e^{b-a}}{-1} + \frac{e^{-a}}{2}\,\frac{1-e^{-b+a}}{+1}$$

$$= \frac{e^a - e^b}{-2} + \frac{e^{-a}-e^{-b}}{2} = \frac{e^b - e^a + e^{-a} - e^{-b}}{2}$$

$$= \frac{e^b - e^{-b}}{2} - \frac{e^a - e^{-a}}{2}$$

$$\int_a^b \cosh x\,dx = \sinh b - \sinh a \quad \text{Ans}$$

Q No. 8 :- $\int_0^{\pi/2} \cos x\,dx$

$\int_a^b f(x)\,dx \Rightarrow f(x) = \cos x$

Length of each Sub-Interval $= \Delta x = \frac{b-a}{n}$

$P = [a, a+\Delta x, a+2\Delta x, a+3\Delta x, \ldots, a+(n-1)\Delta x, a+n\Delta x]$

Sub-Interval:-

$[a, a+\Delta x], [a+\Delta x, a+2\Delta x], [a+2\Delta x, a+3\Delta x], \ldots, [a+(n-1)\Delta x, a+n\Delta x]$.

BY Reimenn Sum :-

$$S(P,f) = \Delta x\{f(a) + f(a+\Delta x) + f(a+2\Delta x) + \cdots + \text{to } n\text{-terms}\}$$

$$= \Delta x\{\cos a + \cos(a+\Delta x) + \cos(a+2\Delta x) + \cdots + \text{to } n\text{-terms}\}$$

So,

$$S(P,f) = \Delta x\left\{\frac{\cos\left(a + \frac{(n-1)}{2}\Delta x\right)\sin\frac{n\Delta x}{2}}{\sin\frac{\Delta x}{2}}\right\}$$

$$= \Delta x\left\{\frac{\cos\left(a + \left(\frac{b-a}{\Delta x}-1\right)/2\cdot\Delta x\right)\sin\left(\frac{b-a}{\Delta x}\right)\cdot\frac{\Delta x}{2}}{\sin\frac{\Delta x}{2}}\right\} \because n = \frac{b-a}{\Delta x}$$

$$S(P,f) = \Delta x\left\{\frac{\cos(2a+b-a-\Delta x)/2\cdot\sin(b-a/2)}{\sin\frac{\Delta x}{2}}\right\}$$

Taking $\lim_{n\to\infty}$ as $\Delta x \to 0$

$$\lim_{n\to\infty} S(P,f) = \frac{\lim_{\Delta x\to 0} 2\cos(a+b-\Delta x)/2\,\sin(b-a)/2}{\lim_{\Delta x\to 0}\frac{\sin \Delta x/2}{\Delta x/2}}$$

$\int_a^b \cos x\,dx = 2\cos\left(\frac{a+b}{2}\right)\cdot \sin\left(\frac{b-a}{2}\right)$

$= \sin\left(\frac{a+b+b-a}{2}\right) - \sin\left(\frac{a+b-b+a}{2}\right)$

$= \sin\left(\frac{2b}{2}\right) - \sin\left(\frac{2a}{2}\right)$

$= \sin b - \sin a$

Now,

$\int_a^b \cos x\,dx = \int_0^{\pi/2} \cos x\,dx = \sin\left(\frac{\pi}{2}\right) - \sin(0)$

$\int_0^{\pi/2} \cos x\,dx = 1$ Ans

Q No. 9:- $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n}$

Let,

$y = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n}$

$\lim_{n\to\infty} y = \lim_{n\to\infty} \frac{1}{n}\left\{\frac{n}{n+1} + \frac{n}{n+2} + \frac{n}{n+3} + \cdots + \frac{n}{n+n}\right\}$

$= \lim_{n\to\infty} \frac{1}{n}\left\{\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \frac{1}{1+\frac{3}{n}} + \cdots + \frac{1}{1+\frac{n}{n}}\right\}$

$= \int_0^1 \frac{1}{1+x}\,dx = \ln(1+x)\Big|_0^1$

$= \ln(1+1) - \ln(1+0) = \ln 2 - \ln 1$

$= \ln 2$ Ans

Q No. 10:- $\frac{n}{n^2}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+(n-1)^2}$

Let,

$$y=\frac{n}{n^2}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots+\frac{n}{n^2+(n-1)^2}$$

$$y=\frac{1}{n}\left\{\frac{n^2}{n^2}+\frac{n^2}{n^2+1^2}+\frac{n^2}{n^2+2^2}+\frac{n^2}{n^2+3^2}+\cdots+\frac{n^2}{n^2+(n-1)^2}\right\}$$

$$y=\frac{1}{n}\left\{\frac{1}{1+(\frac{0}{n})^2}+\frac{1}{1+(\frac{1}{n})^2}+\frac{1}{1+(\frac{2}{n})^2}+\frac{1}{1+(\frac{3}{n})^2}+\cdots+\frac{1}{1+(\frac{n-1}{n})^2}\right\}$$

$$\lim_{n\to\infty}y=\lim_{n\to\infty}\frac{1}{n}\left\{\frac{1}{1+(\frac{0}{n})^2}+\frac{1}{1+(\frac{1}{n})^2}+\frac{1}{1+(\frac{2}{n})^2}+\frac{1}{1+(\frac{3}{n})^2}+\cdots+\frac{1}{1+(\frac{n-1}{n})^2}\right\}$$

$$\lim_{n\to\infty}y=\int_0^1\left(\frac{1}{1+x^2}\right)dx=\operatorname{Tan}^{-1}x\Big|_0^1=\operatorname{Tan}^{-1}(1)-\operatorname{Tan}^{-1}(0)$$

$$=\frac{\pi}{4}-0=\frac{\pi}{4}\ \text{Ans}$$

Q No. 11:- $\frac{n}{(n+1)^2}+\frac{n}{(n+2)^2}+\cdots+\frac{n}{(n+n)^2}$

Let,

$$y=\frac{n}{(n+1)^2}+\frac{n}{(n+2)^2}+\frac{n}{(n+3)^2}+\cdots+\frac{n}{(n+n)^2}$$

$$y=\frac{1}{n}\left\{\frac{n^2}{(n+1)^2}+\frac{n^2}{(n+2)^2}+\frac{n^2}{(n+3)^2}+\cdots+\frac{n^2}{(n+n)^2}\right\}$$

$$\lim_{n\to\infty}y=\lim_{n\to\infty}\frac{1}{n}\left\{\frac{1}{(1+\frac{1}{n})^2}+\frac{1}{(1+\frac{2}{n})^2}+\frac{1}{(1+\frac{3}{n})^2}+\cdots+\frac{1}{(1+\frac{n}{n})^2}\right\}$$

$$=\int_0^1\left(\frac{1}{(1+x)^2}\right)dx=\int_0^1(1+x)^{-2}dx$$

$$=\frac{(1+x)^{-1}}{-1}\Big|_0^1=\frac{-1}{(1+x)}\Big|_0^1=\frac{-1}{(1+1)}+\frac{1}{(1+0)}$$

$$=\frac{-1}{2}+1=\frac{-1+2}{2}=\frac{1}{2}\ \text{Ans}$$

Q No. 12:- $\frac{1}{n\sqrt{n}}\left[\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3}+\cdots+\sqrt{n+n}\right]$

Let,

$$y = \frac{1}{n\sqrt{n}}\left[\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3}+\cdots+\sqrt{n+n}\right]$$

$$= \frac{1}{n}\left[\sqrt{\frac{n+1}{n}}+\sqrt{\frac{n+2}{n}}+\sqrt{\frac{n+3}{n}}+\cdots+\sqrt{\frac{n+n}{n}}\right]$$

$$= \frac{1}{n}\left[\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\sqrt{1+\frac{3}{n}}+\cdots+\sqrt{1+\frac{n}{n}}\right]$$

$$\lim_{n\to\infty} y = \lim_{n\to\infty}\frac{1}{n}\left[\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\sqrt{1+\frac{3}{n}}+\cdots+\sqrt{1+\frac{n}{n}}\right]$$

$$= \int_0^1 \sqrt{1+x}\,dx = \left.\frac{(1+x)^{3/2}}{3/2}\right|_0^1$$

$$= \frac{2}{3}\left[(1+1)^{3/2} - (1+0)^{3/2}\right]$$

$$= \frac{2}{3}\left[\sqrt{8}-1\right] = \frac{2}{3}\left[2\sqrt{2}-1\right] \text{ Ans}$$

Q No. 13:- $\frac{1}{n}+\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}$

Let,

$$y = \frac{1}{n}+\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\frac{1}{\sqrt{n^2-3^2}}+\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}$$

$$y = \frac{1}{n}\left\{\frac{1}{\sqrt{1-(\frac{0}{n})^2}}+\frac{1}{\sqrt{1-(\frac{1}{n})^2}}+\frac{1}{\sqrt{1-(\frac{2}{n})^2}}+\frac{1}{\sqrt{1-(\frac{3}{n})^2}}+\cdots+\frac{1}{\sqrt{1-(\frac{n-1}{n})^2}}\right\}$$

$$\lim_{n\to\infty} y = \lim_{n\to\infty}\frac{1}{n}\left\{\frac{1}{\sqrt{1-(\frac{0}{n})^2}}+\frac{1}{\sqrt{1-(\frac{1}{n})^2}}+\frac{1}{\sqrt{1-(\frac{2}{n})^2}}+\cdots+\frac{1}{\sqrt{1-(\frac{n-1}{n})^2}}\right\}$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \left.\sin^{-1}x\right|_0^1 = \sin^{-1}(1) - \sin^{-1}(0)$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2} \text{ Ans}$$