Ex # 5.2

Q No. 1:- $\int_0^6 f(x)\,dx \; ; \; = \begin{cases} x^2 & ; x<2 \\ 3x-2 & ; x \geq 2 \end{cases}$

$= \int_0^2 f(x)\,dx + \int_2^6 f(x)\,dx$

$= \int_0^2 x^2\,dx + \int_2^6 (3x-2)\,dx = \frac{x^3}{3}\Big|_0^2 + \left(\frac{3x^2}{2} - 2x\right)\Big|_2^6$

$= \frac{1}{3}(8-0) + (54-12) - (6-4)$

$= \frac{8}{3} + 42 - 2 = \frac{8}{3} + 40 = \frac{128}{3}$ Ans

Q No. 2:- $\int_{-1}^5 |x-2|\,dx$

$= \int_{-1}^2 |x-2|\,dx + \int_2^5 |x-2|\,dx$

$= \int_{-1}^2 (2-x)\,dx + \int_2^5 (x-2)\,dx = \left(2x - \frac{x^2}{2}\right)\Big|_{-1}^2 + \left(\frac{x^2}{2} - 2x\right)\Big|_2^5$

$= (4-2) - \left(-2-\frac{1}{2}\right) + \left(\frac{25}{2} - 10\right) - (2-4)$

$= 2 + \frac{5}{2} + \frac{5}{2} + 2 = \frac{18}{2} = 9$ Ans

Q No. 3:- $\int_0^{3\pi/4} |\cos x|\,dx$

$= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{3\pi/4} (-\cos x)\,dx$

$= \sin x\Big|_0^{\pi/2} - \sin x\Big|_{\pi/2}^{3\pi/4} = \left(\sin\frac{\pi}{2} - \sin 0\right) - \left(\sin\frac{3\pi}{4} - \sin\frac{\pi}{2}\right)$

$= (1-0) - \left(\frac{1}{\sqrt{2}} - 1\right) = 1 - \frac{1}{\sqrt{2}} + 1 = 2 - \frac{1}{\sqrt{2}}$ Ans

Q No. 4:- $\int_0^{\pi} \cos^{2n+1} x\,dx$

Let $I = \int_0^{\pi} \cos^{2n+1} x\,dx = \int_0^{\pi} \cos^{2n+1}(\pi - x)\,dx$

$= \int_0^{\pi} -\cos^{2n+1} x\,dx = -\int_0^{\pi} \cos^{2n+1} x\,dx$

$I = -I \Rightarrow 2I = 0 \qquad I = 0$

So, $\int_0^{\pi} \cos^{2n+1} x\,dx = 0$

Q No. 5:- $\int_0^{\pi/4} \frac{\sec^2\theta\,dx}{\tan x - \tan\theta}, \; \theta > \pi/4$

$= \sec^2\theta \int_0^{\pi/4} \frac{dx}{\tan x - \tan\theta}$; Put $z = \tan x$, $dx = \frac{dz}{1+z^2}$

and $\tan\theta = a$

Now, $\int \frac{dx}{\tan x - \tan\theta} = \int \frac{dz/1+z^2}{z-a} = \int \frac{dz}{(z-a)(z^2+1)}$

$$= \int \frac{dz}{(z-a)(z^2+1)} \qquad \frac{1}{(z-a)(z^2+1)} = \frac{A}{z-a} + \frac{Bz+C}{z^2+1}$$

$$1 = A(z^2+1) + Bz+C\,(z-a)$$

Put $z = a$

$$1 = A(a^2+1) \quad \Rightarrow A = \frac{1}{a^2+1}$$

$$\Rightarrow 1 = Az^2 + A + Bz^2 - Baz + Cz - Ca$$

$$A + B = 0 \qquad \Rightarrow B = \frac{-1}{a^2+1}$$

$$A - Ca = 1 \qquad \Rightarrow -A + 1 = -Ca \qquad \Rightarrow A - 1 = Ca$$

$$\frac{\cancel{1} - a^2 - \cancel{1}}{a^2+1} = Ca \qquad \Rightarrow C = \frac{-a}{a^2+1}$$

So, $\frac{1}{(z-a)(z^2+1)} = \frac{1}{(z-a)(a^2+1)} + \frac{\frac{-1}{1+a^2}z - \frac{a}{1+a^2}}{z^2+1}$

$$= \int\left(\frac{1}{(1+a^2)(z-a)} - \frac{z+a}{(z^2+1)(1+a^2)}\right)dz$$

$$= \frac{1}{1+a^2}\int \frac{dz}{z-a} - \int \frac{z+a}{z^2+1}\,dz$$

$$= \frac{1}{1+a^2}\left\{\ln|z-a| - \frac{1}{2}\int \frac{2z}{z^2+1}\,dz - a\int \frac{1}{z^2+1}\,dz\right\}$$

$$= \frac{1}{1+a^2}\left\{\ln|z-a| - \frac{1}{2}\ln|z^2+1| - a\tan^{-1}z\right\}$$

$$= \frac{1}{1+\tan^2\theta}\left\{\ln|\tan x - \tan\theta| - \frac{1}{2}\ln|\sec^2 x| - a\tan^{-1}(\tan x)\right\}$$

$$= \cancel{\sec^2\theta}\,\frac{1}{\cancel{\sec^2\theta}}\left\{\ln|\tan x - \tan\theta|\Big|_0^{\pi/4} - \ln|\sec x|\Big|_0^{\pi/4} - x\tan\theta\Big|_0^{\pi/4}\right\}$$

$$= \ln|1 - \tan\theta| - \ln|0 - \tan\theta| - \left(\ln\sec\frac{\pi}{4} - \ln\sec 0\right) - \left(\frac{\pi}{4}\tan\theta - 0\right)$$

$$= \ln|\tan\theta - 1| - \ln|\tan\theta| - \ln\sqrt{2} - \frac{\pi}{4}\tan\theta$$

$$= \ln\left|\frac{\tan\theta - 1}{\tan\theta}\right| - \ln 2^{1/2} - \frac{\pi}{4}\tan\theta.$$

$$= \ln\left|\frac{\sin\theta/\cos\theta - \sin\frac{\pi}{4}/\cos\frac{\pi}{4}}{\sin\theta/\cos\theta}\right| - \ln 2^{1/2} - \frac{\pi}{4}\tan\theta$$

$$= \ln\left|\frac{\sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4}}{\sin\theta\cdot\cos\frac{\pi}{4}}\right| - \ln 2^{1/2} - \frac{\pi}{4}\tan\theta$$

$$= \ln\left|\frac{\sin(\theta-\pi/4)}{\sin\theta}\right| - \ln\cos\frac{\pi}{4} - \ln 2^{1/2} - \frac{\pi}{4}\tan\theta$$

$$= \ln\left|\frac{\sin(\theta-\pi/4)}{\sin\theta}\right| + \cancel{\ln 2^{1/2}} - \cancel{\ln 2^{1/2}} - \frac{\pi}{4}\tan\theta$$

$$= \ln\left|\frac{\sin(\theta-\pi/4)}{\sin\theta}\right| - \frac{\pi}{4}\tan\theta \quad \text{Ans}$$

Q No. 6:- $\int_0^{\pi/2} \tan x \cdot \ln(\sin x)\,dx$

$$= \int_0^{\pi/2} \frac{\sin x}{\cos x}\ln(\sin x)\,dx = \int_0^{\pi/2} \frac{\ln(\sqrt{1-\cos^2 x})(\sin x)}{\cos x}\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \ln\frac{(1-\cos^2 x)}{\cos x}\sin x\,dx = \frac{1}{2}\int_0^{\pi/2} \ln(1-\cos^2 x)\cdot\frac{\sin x}{\cos x}\,dx$$

Put $z = \cos x$; $-dz = \sin x\,dx$

When $x \to 0$, $z \to 1$; when $x \to \frac{\pi}{2}$, $z \to 0$.

$$= \frac{1}{2}\int_1^0 \frac{\ln(1-z^2)}{z}(-dz) = \frac{1}{2}\int_0^1 \frac{\ln(1-z^2)}{z}\,dz$$

We know that;

$$\ln(1-z^2) = \left(-z^2 - \frac{z^4}{2} - \frac{z^6}{3} - \frac{z^8}{4} - \cdots\right)$$

So $$= \frac{1}{2}\int_0^1 \frac{1}{z}\left(-z^2 - \frac{z^4}{2} - \frac{z^6}{3} - \frac{z^8}{4} \cdots\right)$$

$$= -\frac{1}{2}\int_0^1 \left(z + \frac{z^3}{2} + \frac{z^5}{3} + \frac{z^7}{4} \cdots\right)$$

$$= -\frac{1}{2}\left[\frac{z^2}{2} + \frac{z^4}{8} + \frac{z^6}{18} + \frac{z^8}{32} \cdots\right]_0^1$$

$$= -\frac{1}{4}\left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \cdots - 0\right] = -\frac{1}{4}\left(\frac{\pi^2}{6}\right)$$

$$= \frac{-\pi^2}{24} \quad \text{Ans}$$

The rth Subinterval $[x_{r-1}, x_r]$ and its length $x_r - x_{r-1}$ Will both be denoted by Δx_r.

The norm of P as such:-

$$\|P\| = \max_{1 \le r \le n} \Delta x_r$$

Let c_r be any point of $[x_{r-1}, x_r]$; $r = 1, 2, 3, \ldots, n$.

The Expression,

$$(x_1 - x_0) f(c_1) + (x_2 - x_1) f(c_2) + \cdots + (x_r - x_{r-1}) f(c_r) + \cdots$$

$$\cdots + (x_n - x_{n-1}) f(c_n)$$

$$= \sum_{r=1}^{n} (x_r - x_{r-1}) f(c_r)$$

$$= \sum_{r=1}^{n} \Delta x_r \, f(c_r)$$

is called Reimann Sum. This Sum denoted by $S(P, f)$.

$$\int_a^b f(x)\,dx \quad \text{or} \quad \int_a^b f$$

In this case, f is said to be integrable over $[a,b]$

The numbers a and b are called Lower and upper Limits of integration.

Q No. 7:- $\int_0^{2\pi} \frac{dx}{5+3\cos x}$

$$\int_0^{2\pi} \frac{dx}{5+3\cos x} = 2\int_0^{\pi} \frac{dx}{5+3\cos x}$$

Put $z = \tan\frac{x}{2}$, $dx = \frac{2dz}{1+z^2}$

$\cos x = \frac{1-z^2}{1+z^2}$

$$= 2\int_0^{\pi} \frac{2dz/1+z^2}{5+3\left(\frac{1-z^2}{1+z^2}\right)} = 2\int_0^{\pi} \frac{2dz}{5+5z^2+3-3z^2}$$

$$= 2\int \frac{2dz}{2z^2+8} = 2\cdot\frac{\cancel{2}}{\cancel{2}}\int \frac{dz}{z^2+2^2} = 2\cdot\frac{1}{2}\tan^{-1}\frac{z}{2}$$

$$= \cancel{2}\cdot\frac{1}{\cancel{2}}\tan^{-1}\left(\frac{\tan x/2}{2}\right)\Big|_0^{\pi} = \tan^{-1}\left(\frac{\tan\pi/2}{2}\right) - \tan^{-1}\left(\frac{\tan 0}{2}\right)$$

$$= \tan^{-1}(\infty) - 0 = \frac{\pi}{2} \text{ Ans}$$

Q No. 8:- $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$

$$= \int_0^1 \tan^{-1}\frac{2(1-x)-1}{1+(1-x)-(1-x)^2}dx = \int_0^1 \tan^{-1}\frac{2-2x-1}{1+1-x-1+2x-x^2}dx$$

$$= \int_0^1 \tan^{-1}\frac{-2x+1}{1+x-x^2}dx = -\int_0^1 \tan^{-1}\frac{2x-1}{1+x-x^2}dx$$

$$2\int_0^1 \tan^{-1}\frac{2x-1}{1+x-x^2}dx = 0 \Rightarrow \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx = 0 \text{ Ans}$$

Q No. 9:- $\int_{-\pi/4}^{\pi/4} \frac{2x^3-x}{(x^2+1)(x-1)(x+1)}dx$

$$= \int_{-\pi/4}^{0} \frac{2x^3-x}{(x^2+1)(x-1)(x+1)}dx + \int_0^{\pi/4} \frac{2x^3-x}{(x^2+1)(x-1)(x+1)}dx \quad \text{——— (i)}$$

$\Rightarrow$ Put in Ist integer $x = -z$, $dx = -dz$ when $x \to 0$, $z \to 0$

$x \to -\pi/4$, $z \to \pi/4$

$$= \int_0^{\pi/4} \frac{-2z^3+z}{(z^2+1)(-z-1)(-z+1)}-dz = -\int_0^{\pi/4} \frac{-(2z^3-z)}{(z^2+1)\cdot-(z+1)\cdot-(z-1)}-dz$$

By (i):-

$$\Rightarrow -\int_0^{\pi/4} \frac{2x^3-x}{(x^2+1)(x+1)(x-1)}dx + \int_0^{\pi/4} \frac{2x^3-x}{(x^2+1)(x+1)(x-1)}dx$$

$$= 0 \text{ Ans}$$

Q No. 10:- $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$

$$= \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$$

$$= \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}dx - \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$$

$$2\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}dx = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}dx = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

Put $\cos x = t$, $\sin x\,dx = -dt$

when $x \to 0$, $t \to 1$

$x \to \pi$, $t \to -1$

$$= \pi\int_1^{-1} \frac{-dt}{1+t^2} = -\pi\int_{-1}^{1} \frac{-dt}{1+t^2}$$

$$= \pi\left\{\operatorname{Tan}^{-1}(t)\right\}\Big|_{-1}^{1} = \pi\left\{\operatorname{Tan}^{-1}(1) - \operatorname{Tan}^{-1}(-1)\right\}$$

$$= \pi\left\{\pi/4 + \frac{\pi}{4}\right\} = \pi\left(2\frac{\pi}{4}\right) = \pi^2/2$$

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4} \text{ Ans}$$

Q No. 11:- $\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(3+x)}}\,dx$

Put $9-x = 3+y \Rightarrow dx = -dy$

when $x \to 2$, $y \to 4$

when $x \to 4$, $y \to 2$

$$I = \int_4^2 \frac{\sqrt{\ln(3+y)}}{\sqrt{\ln(3+y)}+\sqrt{\ln(9-y)}}\,-dy$$

$$= \int_2^4 \frac{\sqrt{\ln(3+x)}}{\sqrt{\ln(3+x)}+\sqrt{\ln(9-x)}}\,dx = I$$

$$2I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(3+x)}}\,dx + \int_2^4 \frac{\sqrt{\ln(3+x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(3+x)}}\,dx$$

$$= \int_2^4 \frac{\sqrt{\ln(9-x)}+\sqrt{\ln(3+x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(3+x)}}\,dx \Rightarrow I = 1/2\cdot(x)\Big|_2^4 \Rightarrow \frac{1}{2}\cdot 2$$

$$I = 1 \text{ Ans}$$

Q No. 12:- $\int_0^{\pi/2} \ln(\tan x)\,dx = 0$

$$= \int_0^{\pi/2} \ln\left(\frac{\sin x}{\cos x}\right)dx = \int_0^{\pi/2} (\ln \sin x - \ln \cos x)\,dx$$

$$= \int_0^{\pi/2} \ln \sin x\,dx - \int_0^{\pi/2} \ln \cos x\,dx$$

$$= \int_0^{\pi/2} \ln \sin x\,dx - \int_0^{\pi/2} \ln\left(\cos\left(\frac{\pi}{2} - x\right)\right)dx$$

$$= \int_0^{\pi/2} \ln \cancel{\sin x}\,dx - \int_0^{\pi/2} \ln \cancel{\sin x}\,dx$$

$$= 0 \quad \text{Ans}$$

Q No. 13:- $\int_0^{\pi/2} \sin 2x \ln(\tan x)\,dx = 0$

$$= \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \ln\left(\tan\left(\frac{\pi}{2} - x\right)\right)dx$$

$$= \int_0^{\pi/2} \sin 2x \cdot \ln \cot x\,dx = \int_0^{\pi/2} \sin 2x \cdot \ln\left(\frac{1}{\tan x}\right)dx$$

$$= \int_0^{\pi/2} \sin 2x \cdot \ln(\tan x)^{-1}\,dx = -\int_0^{\pi/2} \sin 2x \ln(\tan x)\,dx$$

$$2\int_0^{\pi/2} \sin 2x \ln(\tan x)\,dx = 0$$

$$\int_0^{\pi/2} \sin 2x \ln(\tan x)\,dx = 0 \quad \text{Ans}$$

Q No. 14:- $\int_0^{\pi} \frac{x \tan x}{\sec x + \cos x}\,dx$

$$= \int_0^{\pi} \frac{x \cdot \sin x/\cos x}{1 + \cos^2 x/\cos x}\,dx = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)}\,dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x}\,dx = \int_0^{\pi} \frac{\pi}{1+\cos^2 x}\sin x - \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\,dx$$

By Q No. 10:- $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\,dx = \frac{\pi^2}{4}$ Ans

Q No. 15:- $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \pi/4$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin(\frac{\pi}{2}-x)}}{\sqrt{\sin(\frac{\pi}{2}-x)}+\sqrt{\cos(\frac{\pi}{2}-x)}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = 2I$$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{\cancel{\sqrt{\sin x}+\sqrt{\cos x}}}{\cancel{\sqrt{\sin x}+\sqrt{\cos x}}}\,dx = x\Big|_0^{\pi/2} = \left(\frac{\pi}{2}-0\right)$$

$$2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4} \text{ Ans}$$

Q No. 16:- $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x}\,dx = \frac{1}{\sqrt{2}} \ln(\sqrt{2}+1)$

$$= \int_0^{\pi/2} \frac{\sin^2(\pi/2-x)}{\sin(\frac{\pi}{2}-x)+\cos(\frac{\pi}{2}-x)}\,dx = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x+\sin x}\,dx$$

$$2I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x}\,dx + \int_0^{\pi/2} \frac{\cos^2 x}{\sin x+\cos x}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin^2 x+\cos^2 x}{\sin x+\cos x}\,dx = \int_0^{\pi/2} \frac{dx}{\sin x+\cos x}$$

Put $z = \tan x/2$, $dx = \frac{2dz}{1+z^2}$

when $x \to 0$, $z \to 0$

$x \to \frac{\pi}{2}$, $z \to 1$

$$2I = \int_0^1 \frac{2dz/\cancel{1+z^2}}{2z+1-z^2/\cancel{1+z^2}} = 2\int_0^1 \frac{dz}{2z+1-z^2} = -2\int_0^1 \frac{dz}{z^2-2z-1}$$

$$= -2\int_0^1 \frac{dz}{z^2-2z+1-1-1} = -2\int_0^1 \frac{dz}{(z-1)^2-(\sqrt{2})^2} = 2\int_0^1 \frac{dz}{(\sqrt{2})^2-(z-1)^2}$$

$$= \cancel{2}\cdot\frac{1}{\cancel{2}\cdot\sqrt{2}} \ln\left|\frac{\sqrt{2}+(z-1)}{\sqrt{2}-(z-1)}\right|\Bigg|_0^1 = \frac{1}{\sqrt{2}}\left\{\ln 1 - \ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\}$$

$$2I = \frac{1}{\sqrt{2}} \ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^{-1} = \frac{1}{\sqrt{2}} \ln\left[\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\times\left(\frac{\sqrt{2}+1}{\sqrt{2}+1}\right)\right]$$

$$= \frac{1}{\sqrt{2}} \ln(\sqrt{2}+1)^2 = \frac{2}{\sqrt{2}} \ln(\sqrt{2}+1)$$

$$\cancel{2} I = \frac{\cancel{2}}{\sqrt{2}} \ln(\sqrt{2}+1)$$

$$I = \frac{1}{\sqrt{2}} \ln(\sqrt{2}+1) \text{ Ans}$$

Q No. 17:- $\int_0^{\pi} \frac{x\,dx}{1+\sin x} = \pi$

$$\int_0^{\pi} \frac{x}{1+\sin x}\,dx = \int_0^{\pi} \frac{(\pi-x)}{1+\sin(\pi-x)}\,dx = \int_0^{\pi} \frac{(\pi-x)}{1+\sin x}\,dx$$

$$2I = \int_0^{\pi} \frac{\pi}{1+\sin x}\,dx = \pi\int_0^{\pi} \left(\frac{1}{1+\sin x} \cdot \frac{1-\sin x}{1-\sin x}\right) dx$$

$$= \pi\int_0^{\pi} \frac{1-\sin x}{\cos^2 x}\,dx = \pi\int_0^{\pi} (\sec^2 x - \sec x \tan x)\,dx$$

$$= \pi\left\{\int_0^{\pi} \sec^2 x\,dx - \int_0^{\pi} \sec x \tan x\,dx\right\}$$

$$2I = \pi\left\{\tan x\Big|_0^{\pi} - \sec x\Big|_0^{\pi}\right\} = \pi\left\{(\tan(\pi) - 0) - (\sec(\pi) - \sec(0))\right\}$$

$$= \pi\left\{0 - (-1-1)\right\} = \pi(2) = 2\pi$$

$$I = \pi \text{ Ans}$$

Q No. 18:- $\int_0^{\pi/2} \left(\frac{\theta}{\sin\theta}\right)^2 d\theta = \pi \ln 2$

$$\int_0^{\pi/2} \theta^2 \csc^2\theta\,d\theta = \theta^2 \int_0^{\pi/2} \csc^2\theta\,dx - \int_0\left(\frac{d}{d\theta}\theta^2 \cdot \int \csc^2\theta\,d\theta\right) d\theta$$

$$= -\theta^2 \cot\theta - \int 2\theta(-\cot\theta)\,d\theta = -\theta^2\cot\theta + 2\left[\int \theta \cdot \cot\theta\,d\theta\right]$$

$$= -\theta^2\cot\theta + 2\left[\theta \ln\sin\theta - \cancel{1}\int \ln\sin\theta\right] d\theta$$

$$= -\theta^2\cot\theta + 2\theta \ln \overset{\sin}{\theta} - 2\int \ln\sin\theta\,d\theta$$

$$= -\theta^2\cot\theta\Big|_0^{\pi/2} + 2\theta\ln\sin\theta\Big|_0^{\pi/2} - 2\int^{\pi/2} \ln\sin\theta\,d\theta$$

$$= 0 + 0 - \cancel{2}\left(-\frac{\pi}{\cancel{2}}\ln 2\right) = \pi \ln 2$$

$$= \pi \ln 2 \text{ Ans}$$

QNo 19:- $\int_0^{\pi/2} \ln(\tan\theta + \cot\theta)\, d\theta = \pi \ln 2$

$$\int_0^{\pi/2} \ln\left(\tan\theta + \frac{1}{\tan\theta}\right) d\theta = \int_0^{\pi/2} \ln\left(\frac{1+\tan^2\theta}{\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \ln\left(\frac{\sec^2\theta}{\tan\theta}\right) d\theta = \int_0^{\pi/2} \ln\left(\frac{d\theta}{\sin\theta\cos\theta}\right)$$

$$= -\int_0^{\pi/2} \ln(\sin\theta\cos\theta)\, d\theta = -\int_0^{\pi/2} \left[\ln\sin\theta\, d\theta + \ln\cos\theta\, d\theta\right]$$

$$= -\int_0^{\pi/2} \ln\sin\theta\, d\theta - \int_0^{\pi/2} \ln\cos\theta\, d\theta$$

$$= -\int_0^{\pi/2} \ln\sin\theta\, d\theta - \int_0^{\pi/2} \ln\left(\cos\left(\frac{\pi}{2} - \theta\right)\right) d\theta$$

$$= -\int_0^{\pi/2} \ln\sin\theta\, d\theta - \int_0^{\pi/2} \ln\sin\theta\, d\theta$$

$$= -2\int_0^{\pi/2} \ln\sin\theta\, d\theta = -\cancel{2}\left(-\pi/\cancel{2}\ \ln 2\right)$$

$$= \pi \ln 2 \quad \text{Ans}$$

QNo. 20:- $\int_0^{\pi} \ln(\sin x) \cdot x\, dx = \frac{\pi^2}{2} \ln\left(\frac{1}{2}\right)$

$$\int_0^{\pi} (\pi - x) \cdot \ln(\sin(\pi - x))\, dx = \int_0^{\pi} (\pi - x) \ln\sin x\, dx$$

$$= \pi\int_0^{\pi} \ln\sin x\, dx - x\int_0^{\pi} \ln\sin x\, dx$$

$$\cancel{2}\int_0^{\pi} x\ln(\sin x)\, dx = \pi \cdot \cancel{2}\int_0^{\pi/2} \ln\sin x\, dx$$

$$= \pi\left(\frac{\pi}{2} \ln\left(\frac{1}{2}\right)\right)$$

$$= \frac{\pi^2}{2} \ln\left(\frac{1}{2}\right) \quad \text{Ans}$$

QNo. 21:- $\int_0^{1} \frac{\ln(1+x)}{1+x^2}\, dx$

Put $x = \tan\theta$ When $x \to 0$ as $\theta \to 0$

$dx = \sec^2\theta\, d\theta$ $x \to 1$ as $\theta \to \frac{\pi}{4}$

$$= \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta} \sec^2\theta \, d\theta = \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{\cancel{\sec^2\theta}} \cancel{\sec^2\theta} \, d\theta$$

$$= \int_0^{\pi/4} \ln(1+\tan(\tfrac{\pi}{4}-\theta)) \, d\theta$$

$$= \int_0^{\pi/4} \ln\left(1 + \frac{\tan\pi/4 - \tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \ln\left(\frac{1+\cancel{\tan\theta}+1-\cancel{\tan\theta}}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\theta}\right) d\theta = \int_0^{\pi/4} \ln 2 - \ln(1+\tan\theta) d\theta$$

$$2\int_0^{\pi/4} (1+\tan\theta) \, d\theta = \ln 2 \int_0^{\pi/4} 1 \, d\theta$$

$$= \ln 2 \cdot \theta \Big|_0^{\pi/4}$$

$$= \ln 2 \left(\tfrac{\pi}{4} - 0\right)$$

$$= \ln 2 \left(\tfrac{\pi}{4}\right)$$

$$\int_0^{\pi/4} (1+\tan\theta) \, d\theta = \ln 2 \cdot \frac{\pi}{8}$$

$$\int_0^1 \frac{\ln(1+x)}{1+x^2} dx = \ln 2 \cdot \frac{\pi}{8} \quad \text{Ans}$$

Q No. 22:- $\int_0^{\pi/2} \sin x \ln(\sin x) dx = \ln\left(\frac{2}{e}\right)$

$$= \int_0^{\pi/2} \ln\left(\sqrt{1-\cos^2 x}\right) \sin x \, dx$$

$$= \int_0^{\pi/2} \frac{1}{2} \ln(1-\cos^2 x) \sin x \, dx$$

Put $\cos x = t$, $\sin x dx = -dt$

when $x \to 0$ as $t \to 1$; when $x \to \frac{\pi}{2}$ as $t \to 0$

$$= \frac{1}{2}\int_1^0 \ln(1-t^2) \, -dt$$

$$= \frac{1}{2}\int_0^1 \ln(1-t^2) \, dt$$

$$= \frac{1}{2}\int_0^1 \left[-t^2 - \frac{t^4}{2} - \frac{t^6}{3} - \frac{t^8}{4} \cdots\cdots\right] dt$$

$$= -\frac{1}{2}\left[\frac{t^3}{3} + \frac{t^5}{10} + \frac{t^7}{21} + \frac{t^9}{36} + \cdots\right]\Big|_0^1$$

$$= -\frac{1}{2}\left(\frac{1}{3} + \frac{1}{10} + \frac{1}{21} + \frac{1}{36} + \cdots - 0\right)$$

$$= -\left(\frac{1}{2\cdot 3} + \frac{1}{2\cdot 10} + \frac{1}{2\cdot 21} + \frac{1}{2\cdot 36} + \cdots\right)$$

$$= -\left(\frac{1}{2\cdot 3} + \frac{1}{4\cdot 5} + \frac{1}{6\cdot 7} + \frac{1}{8\cdot 9} + \cdots\right)$$

$$= -\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots\right)$$

$$= +\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} \cdots\cdots - 1\right)$$

$$= \ln[(1+1)] - [1] = \ln 2 - 1 = \ln 2 - \ln e$$

$$= \ln\left(\frac{2}{e}\right) \text{ Ans}$$

Q No. 23:- $\int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\, dx = \pi/4$

$$I = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\, dx \quad \text{---(i)}$$

$$= \int_0^{\pi/2} \frac{\cos(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}\, dx = I = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}\, dx \quad \text{--- (ii)}$$

Adding (i) and (ii)

$$2I = \int_0^{\pi/2} \frac{\cancel{\sin x + \cos x}}{\cancel{\sin x + \cos x}}\, dx = x\Big|_0^{\pi/2}$$

$$2I = (\pi/2 - 0) = \frac{\pi}{2}$$

$$I = \frac{\pi}{4} \text{ Ans}$$

Q No. 24:- $\int_0^{\pi} \frac{x \sin x}{1+\sin x}\, dx = \frac{\pi^2}{2} - \pi$

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\sin(\pi - x)}\, dx = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\sin x}\, dx$$

$$2\int_0^{\pi} \frac{x \sin x}{1+\sin x}\, dx = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x}\, dx$$

$$\begin{array}{r|l} & 1 \\ 1+\sin x & \sin x \\ & \underline{1+\sin x} \\ & -1 \end{array}$$

$$= \pi\left[\int_0^{\pi}\left(1 - \frac{1}{1+\sin x}\right) dx\right]$$

$$2\int_0^{\pi} \frac{x\sin x}{1+\sin x}\, dx = \pi\left[(\pi - 0) - \int_0^{\pi} \frac{1}{1+\sin x}\, dx\right]$$

$$= \pi\left[\pi - \int_0^{\pi} \frac{1-\sin x}{\cos^2 x}\, dx\right]$$

$$= \pi\left[\pi - \left(\int_0^{\pi} \sec^2 x\, dx - \int_0^{\pi} \sec x \tan x\, dx\right)\right]$$

$$= \pi^2 - \pi\left(\tan x\Big|_0^{\pi} - \sec x\Big|_0^{\pi}\right)$$

$$2\int_0^{\pi} \frac{x\sin x}{1+\sin x}\, dx = \pi^2 - \pi\left(0 - (-1-1)\right)$$

$$= \pi^2 - 2\pi$$

$$\int_0^{\pi} \frac{x\sin x}{1+\sin x}\, dx = \frac{\pi^2}{2} - \pi \quad \text{Ans}$$

Q No. 25:- $\int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x\cos x}\, dx = 0$

$$I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x\cos x}\, dx \quad \text{---(i)} \quad = \int_0^{\pi/2} \frac{\sin(\frac{\pi}{2} - x) - \cos(\frac{\pi}{2} - x)}{1+\sin(\frac{\pi}{2} - x)\cos(\frac{\pi}{2} - x)}\, dx$$

$$I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1+\cos x\sin x} \quad \text{---(ii)}$$

Adding (i) and (ii)

$$2I = \int_0^{\pi/2} \frac{\cancel{\sin x} - \cancel{\cos x} + \cancel{\cos x} - \cancel{\sin x}}{1 + \sin x \cos x}\, dx$$

$$2I = \int_0^{\pi/2} 0\, dx = 0$$

$$I = 0 \text{ Ans}$$

Q No 26:- $\int_0^{\pi/2} \frac{\sin^2 x}{1+\sin x \cos x}\, dx = \frac{\pi}{3\sqrt{3}}$

$$\int_0^{\pi/2} \frac{\sin^2 x}{1+\sin x \cos x}\, dx \text{ —— (i)} = \int_0^{\pi/2} \frac{\sin^2(\pi/2 - x)}{1+\sin(\frac{\pi}{2}-x)\cos(\frac{\pi}{2}-x)}\, dx$$

$$= \int_0^{\pi/2} \frac{\cos^2 x}{1+\sin x \cos x}\, dx \text{ —— (ii)}$$

Adding (i) and (ii)

$$2I = \int_0^{\pi/2} \left(\frac{(\sin^2 x + \cos^2 x)}{1+\sin x \cos x}\right) dx = \int_0^{\pi/2} \frac{1}{1+\sin x \cos x}\, dx$$

$$= \int_0^{\pi/2} \frac{1}{1+\frac{1}{2}\cdot 2\sin x \cos x}\, dx = \int_0^{\pi/2} \frac{1}{1+\frac{1}{2}\sin 2x}\, dx$$

$$= \int_0^{\pi/2} \frac{dx}{1+\frac{1}{\cancel{2}}\frac{\cancel{2}\tan x}{1+\tan^2 x}} \qquad \because \sin 2x = \frac{2\sin x\cos x/\cos^2 x}{\sin^2 x + \cos^2 x/\cos^2 x} = \frac{2\tan x}{1+\tan^2 x}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\, dx}{\tan^2 x + \tan x + 1} \qquad \text{Put } \tan x = t$$

$$dx \sec^2 x = dt$$

$$2I = \int \frac{dt}{t^2+t+1} = \int \frac{dt}{t^2+t+(\frac{1}{2})^2-(\frac{1}{2})^2+1} = \int \frac{dt}{(t+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$= \left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^{\pi/2} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x+1}{\sqrt{3}}\right)\Big|_0^{\pi/2}$$

$$= \frac{2}{\sqrt{3}}\left\{\tan^{-1}(\infty) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right\} = \frac{2}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$$

$$\cancel{2}I = \frac{2}{\sqrt{3}}\left(\frac{3\pi-\pi}{6}\right) = \frac{4\pi}{6\sqrt{3}} = \frac{\cancel{2}\pi}{3\sqrt{3}} \Rightarrow I = \frac{\pi}{3\sqrt{3}} \text{ A}$$

Q No. 27: Show That: $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

Sol:-

Since $f(x) \le g(x)$; $\forall x \in [a,b]$

$f(c_r) \le g(c_r)$; $c_r \in [x_{r-1}, x_r]$

$f(c_r)\,\Delta x_r \le g(c_r)\,\Delta x_r$

$\sum_{r=1}^{n} f(c_r)\,\Delta x_r \le \sum_{r=1}^{n} g(c_r)\,\Delta x_r$

$\lim_{n\to\infty} S(P,f) \le \lim_{n\to\infty} S(P,g)$

$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

Hence Proved