Therefore $$\lim_{n\to\infty} \frac{\int_0^{\pi/2} \sin^{2n} x \; dx}{\int_0^{\pi/2} \sin^{2n+1} x \, dx} = 1, \tag{7}$$

by the Sandwiching Theorem 1.32 (v).

Taking limits of both sides of (5) as $n \to \infty$ we have

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6 \cdots (2n)(2n)}{1\cdot 2\cdot 3\cdot 5\cdot 5\cdot 7 \cdots (2n-1)(2n+1)} \times \lim_{n\to\infty} \frac{\int_0^{\pi/2} \sin^{2n} x \; dx}{\int_0^{\pi/2} \sin^{2n+1} x \; dx}$$

$$= \lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}, \text{ using (7).}$$

This is known as **Wallis' Product Formula** for $\dfrac{\pi}{2}$.

Exercise Set 5.4

Evaluate (Problems 1 – 21):

1. $\displaystyle\int \frac{\sec^4 x}{\tan^5 x}\, dx$

2. $\displaystyle\int \sin^2 x \cos^4 x \; dx$

3. $\displaystyle\int \sin^6 x \cos^2 x \; dx$

4. $\displaystyle\int \sin^{1/2} x \cos^3 x \; dx$

5. $\displaystyle\int \sec^2 x \csc^3 x \; dx$

6. $\displaystyle\int \tan^3 x \sec^5 x \; dx$

7. $\displaystyle\int \cot^5 x \csc^4 x \; dx$

8. $\displaystyle\int \frac{\sin^2 x}{\cos^5 x}\, dx$

9. $\displaystyle\int_{\pi/4}^{\pi/2} \cot^4 x \; dx$

10. $\displaystyle\int_{\pi/4}^{\pi/2} \cot^3 x \csc^3 x \; dx$

11. $\displaystyle\int_0^{\pi/2} \tan^5\left(\frac{x}{2}\right) dx$

12. $\displaystyle\int_0^{a} (a^2 - x^2)^{5/2}\, dx$

13. $\displaystyle\int_0^{\pi} \frac{\sin^4 x}{(1+\cos x)^2}\, dx$

14. $\displaystyle\int_0^{\pi/4} \sin^4 2x \; dx$

15. $\displaystyle\int_0^{\pi/2} \sin^6 3x\ dx$

16. $\displaystyle\int_0^{\pi/8} \sin^5 4x \cos^4 4x\ dx$

17. $\displaystyle\int_0^{\pi/4} \cos^2 2x\ dx$

18. $\displaystyle\int_0^{\pi/6} \cos^3 3x\ dx$

19. $\displaystyle\int_0^{\pi/2} \sin^2 6x \cos^4 3x\ dx$

20. $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\cos^2 x}{\sin x}\ dx$

21. $\displaystyle\int_0^1 \frac{x^6\, dx}{\sqrt{1-x^2}}$.

22. Show that

$$\int \sec^{2n+1} x\ dx = \frac{\sec^{2n-1} x \tan x}{2n} + \left(1 - \frac{1}{2n}\right) \int \sec^{2n-1} x\ dx.$$

23. Obtain a reduction formula for $\displaystyle\int \frac{dx}{(a^2+x^2)^n}$, where n is an integer. Show that $\displaystyle\int_0^{\infty} \frac{dx}{(1+x^2)^5} = \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\frac{\pi}{2}$.

24. If I_n denotes $\displaystyle\int_0^1 x^p(1-x^q)^n\, dx$, where p, q and n are positive, prove that $(qn+p+1)\, I_n = qn\, I_{n-1}$. Evaluate I_n when n is a positive integer.

25. Obtain a reduction formula for $\displaystyle\int \frac{x^n}{\sqrt{1-x^2}}\, dx$ and hence evaluate $\displaystyle\int \frac{x^3}{\sqrt{1-x^2}}\, dx$.

26. Calculate the value of $\displaystyle\int_0^{2a} x^n \sqrt{2ax - x^2}\ dx$, n being a positive integer.

Hence or otherwise calculate the values of

(i) $\displaystyle\int_0^{2a} x\sqrt{2ax - x^2}\ dx$ (ii) $\displaystyle\int_0^{2a} x^4\sqrt{2ax - x^2}\ dx$

27. If $I_n = \int x^n(a^2 - x^2)^{1/2}\, dx$, prove that

$$I_n = -\frac{x^{n-1}(a^2 - x^2)^{3/2}}{n+2} + \frac{n-1}{n+2}\, a^2 I_{n-2}$$

Hence evaluate $\int_0^a x^4\sqrt{a^2 - x^2}\, dx$.

28. Prove that

$$\int x^m (\ln x)^n\, dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m (\ln x)^{n-1}\, dx.$$

Hence calculate

(i) $\int x^m (\ln x)^3\, dx$ (ii) $\int_0^1 x^m (\ln x)^n\, dx$

29. Prove that

$$\int_0^{\pi/2} \cos^m x \sin nx\, dx = \frac{1}{m+n} + \frac{m}{m+n}\int_0^{\pi/2} \cos^{m-1} x \sin (n-1)x\, dx$$

Hence evaluate $\int_0^{\pi/2} \cos^6 x \sin 3x\, dx$.

30. Find a reduction formula for $\int \frac{x^n}{\sqrt{ax^2 + 2bx + c}}\, dx$.

Numerical Integration

In Chapter 4 we studied various methods of evaluating antiderivatives of certain functions. In order to evaluate definite integrals, the Fundamental Theorem of Integral Calculus (5.3) is a basic tool. But this theorem fails to deliver if the antiderivative of the integral cannot be found in terms of elementary functions (i.e., functions that can be expressed as a finite combination of algebraic and transcendental functions). For such cases Riemann sums provide an approximation of a definite integral when the number of points in partition is large. In practice this method is seldom used since there are better techniques and formulas which give a more efficient way to approximate such integrals. The methods of approximating definite integrals are called **numerical integration**. In this section, we discuss two such methods.

ALI RAZA
M.Sc. Mathematics (P.U)
Mob. # 0345-7845311

Ex # 5.4

Q No. 1:- $\int \frac{\sec^4 x}{\tan^5 x}\,dx$

$\int \frac{\sec^4 x}{\tan^5 x}\,dx \Rightarrow$ Put $\tan x = z$, $\sec^2 x\,dx = dz$

$$\int \frac{\sec^4 x}{\tan^5 x}\,dx = \int \frac{\sec^2 x}{\tan^5 x}\sec^2 x\,dx = \int \frac{1+z^2}{z^5}\,dz$$

$$= \int\left(\frac{1}{z^5}+\frac{1}{z^3}\right)dz = -\frac{1}{4z^4} - \frac{1}{2z^2}$$

$$\int \frac{\sec^4 x}{\tan^5 x}\,dx = -\frac{1}{4\tan^4 x} - \frac{1}{2\tan^2 x} \quad \text{Ans}$$

Q No. 2:- $\int \sin^2 x \cos^4 x\,dx$

We know that,

$$\int \sin^p x \cos^q x\,dx = \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q}\int \sin^p x \cos^{q-2} x\,dx$$

Put $p=2$, $q=4$

$$\Rightarrow \int \sin^2 x \cos^4 x\,dx = \frac{\sin^3 x \cos^3 x}{6} + \frac{1}{2}\int \sin^2 x \cos^2 x\,dx$$

Now,

$$\int \sin^2 x \cos^2 x\,dx = \frac{\sin^3 x \cos x}{4} + \frac{1}{4}\int \sin^2 x \cos^0 x\,dx$$

$$= \frac{\sin^3 x \cos x}{4} + \frac{1}{4}\int \frac{1-\cos 2x}{2}\,dx$$

$$= \frac{\sin^3 x \cos x}{4} + \frac{x}{8} - \frac{\sin 2x}{8\cdot 2}$$

We have,

$$\int \sin^2 x \cos^4 x\,dx = \frac{\sin^3 x \cos^3 x}{6} + \frac{\sin^3 x \cos x}{8} + \frac{x}{16} - \frac{\sin x \cos x}{16}$$

Ans

Q No. 3:- $\int \sin^6 x \cos^2 x\,dx$

We have reduction formula:-

$$\int \sin^p x \cos^q x\,dx = -\frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2} x \cos^q x\,dx$$

Put $p=6$, $q=2$ Then,

$$I=\int \sin^6 x \cos^2 x\,dx = -\frac{\sin^5 x \cos^3 x}{8}+\frac{5}{8}\int \sin^4 x \cos^2 x\,dx$$

$$\Rightarrow \int \sin^4 x \cos^2 x\,dx = -\frac{1}{6}\sin^3 x \cos^3 x+\frac{1}{2}\int \sin^2 x \cos^2 x\,dx$$

$$\Rightarrow \int \sin^2 x \cos^2 x\,dx = -\frac{1}{4}\sin x \cos^3 x+\frac{1}{4}\int \cos^2 x\,dx$$

$$= -\frac{1}{4}\sin x \cos^3 x+\frac{1}{4}\int \frac{1+\cos 2x}{2}\,dx$$

$$= -\frac{1}{4}\sin x \cos^3 x+\frac{1}{4}\left(\frac{x}{2}+\frac{\sin 2x}{4}\right)$$

Therefore:-

$$I=\frac{-1}{8}\sin^5 x \cos^3 x+\frac{5}{8}\left[-\frac{1}{6}\sin^3 x \cos^3 x+\frac{1}{2}\left(-\frac{1}{4}\sin x \cos^3 x\right)+\frac{1}{8}\left(\frac{x}{2}+\frac{\sin 2x}{4}\right)\right]$$

$$I=\frac{-1}{8}\sin^5 x \cos^3 x-\frac{5}{48}\sin^3 x \cos^3 x-\frac{5}{64}\sin x \cos^3 x+\frac{5}{64}\left(\frac{x}{2}+\frac{\sin 2x}{4}\right)$$

$$I=-\frac{\sin^5 x \cos^3 x}{8}-\frac{5}{48}\sin^3 x \cos^3 x-\frac{5}{64}\sin x \cos^3 x+\frac{5x}{128}+\frac{5\sin x \cos x}{128}$$

Ans

Q No. 4:- $\int \sin^{1/2} x \cos^3 x\,dx$

$$I=\int \sqrt{\sin x}\,\cos^3 x\,dx$$

Put $\sqrt{\sin x}=z \Rightarrow z^2=\sin x$

$2z\,dz=\cos x\,dx$

$$I=\int z\cdot\left(1-(z^2)^2\right)\cdot 2z\,dz$$

$$=2\int z^2(1-z^4)\,dz = 2\int (z^2-z^6)\,dz$$

$$I=\frac{2}{3}z^3-\frac{2}{7}z^7$$

$$I=\frac{2}{3}\sin^{3/2} x-\frac{2}{7}\sin^{7/2} x$$

Ans

QNo.5:- $\int \sec^2 x \csc^3 x\, dx$

$I = \int \sec^2 x \csc^3 x\, dx$

Integration by Parts:-

$I = \int \csc^3 x \cdot \sec^2 x\, dx$

$= \csc^3 x \cdot \tan x - \int 3\csc^2 x \cdot (-\csc x \cot x) \cdot \tan x\, dx$

$I = \csc^3 x \tan x - \int -3\csc^3 x \cdot \cot x \tan x\, dx$

$I = \csc^3 x \tan x + 3\int \csc^3 x\, dx$

Now,

BY using reduction formula:-

$\int \csc^3 x\, dx = -\frac{1}{2} \csc x \cot x + \frac{1}{2}\int \csc x\, dx$

$= -\frac{1}{2} \csc x \cot x + \frac{1}{2} \ln|\csc x - \cot x|$

I becomes:-

$I = \csc^3 x \tan x + 3\left[-\frac{1}{2} \csc x \cot x + \frac{1}{2} \ln|\csc x - \cot x|\right]$

$I = \csc^3 x \tan x - \frac{3}{2} \csc x \cot x + \frac{3}{2} \ln|\csc x - \cot x|$

Ans

QNo.6:- $\int \tan^3 x \sec^5 x\, dx$

$I = \int \tan^3 x \sec^5 x\, dx$

Put $\sec x = z$, $\sec x \tan x\, dx = dz$

$I = \int \tan^2 x \sec^4 x (\sec x \tan x) dx$

$I = \int (z^2 - 1) z^4\, dz = \int (z^6 - z^4) dz$

$I = \frac{z^7}{7} - \frac{z^5}{5} = \frac{1}{7} \sec^7 x - \frac{1}{5} \sec^5 x$ Ans

QNo.7:- $\int \cot^5 x \csc^4 x\, dx$

$I = -\int \cot^5 x \csc^2 x (-\csc^2 x) dx$

Put $\cot x = z$, $\csc^2 x\, dx = dz$

$I = -\int z^5 (1 + z^2) dz = -\int (z^5 + z^7) dz$

$I = -\frac{z^6}{6} - \frac{z^8}{8} = -\frac{1}{6} \cot^6 x - \frac{1}{8} \cot^8 x$

Q No.8:- $\int \frac{\sin^2 x}{\cos^5 x}\,dx$

$$I = \int \frac{\sin^2 x}{\cos^5 x}\,dx = \int \frac{\sin^2 x}{\cos^2 x}\cdot\frac{1}{\cos^3 x}\,dx$$

$$I = \int \tan^2 x \sec^3 x\,dx = \int (\sec^2 x - 1)\sec^3 x\,dx$$

$$I = \int \sec^5 x\,dx - \int \sec^3 x\,dx$$

Now, By reduction formula

$$\int \sec^5 x\,dx = \frac{\sec^3 x \tan x}{4} + \frac{3}{4}\int \sec^3 x\,dx$$

So,

$$I = \frac{\sec^3 x \tan x}{4} + \frac{3}{4}\int \sec^3 x\,dx - \int \sec^3 x\,dx$$

$$= \frac{\sec^3 x \tan x}{4} - \frac{1}{4}\int \sec^3 x\,dx$$

Again, $\int \sec^3 x\,dx = \frac{\sec x \tan x}{2} + \frac{1}{2}\int \sec x\,dx$

$$= \frac{\sec x \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x|$$

$$I = \frac{\sec^3 x \tan x}{4} - \frac{1}{8}\sec x \tan x - \frac{1}{8}\ln|\sec x + \tan x|$$ Ans

Q No.9:- $\int_{\pi/4}^{\pi/2} \cot^4 x\,dx$

By reduction formula:-

$$\int \cot^4 x\,dx = -\frac{\cot^3 x}{3} - \int \cot^2 x\,dx$$

$$= -\frac{\cot^3 x}{3} - \int (\csc^2 x - 1)\,dx$$

$$= -\frac{\cot^3 x}{3} - (-\cot x) + x$$

$$= -\frac{\cot^3 x}{3} + \cot x + x$$

$$I = \int_{\pi/4}^{\pi/2} \cot^4 x\,dx = \left[-\frac{\cot^3 x}{3} + \cot x + x\right]_{\pi/4}^{\pi/2}$$

$$I = \frac{\pi}{2} - \left[-\frac{1}{3} + 1 + \frac{\pi}{4}\right] = \frac{\pi}{4} - \frac{2}{3}$$ Ans

Q No. 10:- $\int_{\pi/4}^{\pi/2} \cot^3 x \csc^3 x\, dx$

Put $\csc x = z$, $-\csc x \cot x\, dx = dz$

$I = \int_{\pi/4}^{\pi/2} \cot^3 x \csc^3 x\, dx = -\int_{\pi/4}^{\pi/2} (\csc^2 x - 1)\csc^2 x(\csc x \cot x)dx$

$I = -\int_{\pi/4}^{\pi/2} \csc^4 x - \csc^2 x\ (-\csc x \cot x)dx$

$= -\int_{\sqrt{2}}^{1} (z^4 - z^2)dz = \int_1^{\sqrt{2}} z^4\, dz - \int_1^{\sqrt{2}} z^2 dz$

$I = \left[\frac{z^5}{5} - \frac{z^3}{3}\right]_1^{\sqrt{2}} = \left(\frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)$

$= \sqrt{2}\left(\frac{4}{5} - \frac{2}{3}\right) - \left(-\frac{2}{15}\right)$

$I = \frac{2}{15}(\sqrt{2}+1)$ Ans

Q No. 11 :-

$\int_0^{\pi/2} \tan^5\left(\frac{x}{2}\right)dx$

Put $\cancel{\tan}\ \frac{x}{2} = z$, $dx = 2dz$

when $x \to 0$, $z \to 0$

$x \to \pi/2$, $z \to \frac{\pi}{4}$

$I = \int_0^{\pi/2} \tan^5\left(\frac{x}{2}\right)dx = 2\int_0^{\pi/4} \tan^5 z\, dz$

Now, $\int \tan^5 z\, dz = \frac{\tan^4 z}{4} - \int \tan^3 z\, dz$

$= \frac{\tan^4 z}{4} - \left[\frac{\tan^2 z}{2} - \int \tan z\, dz\right]$

$= \frac{\tan^4 z}{4} - \frac{\tan^2 z}{2} + \ln|\sec z|$

So,

$I = 2\left[\frac{\tan^4 z}{4} - \frac{\tan^2 z}{2} + \ln|\sec z|\right]_0^{\pi/4}$

$I = 2\left[\left(\frac{1}{4} - \frac{1}{2} + \ln\sqrt{2}\right) - 0\right]$

$I = 2\left(-\frac{1}{4}\right) + 2\log\sqrt{2} = -\frac{1}{2} + \ln 2$ Ans

Q No. 12:-

$\int_0^{\pi} \frac{\sin^4 x}{(1+\cos x)^2}\, dx.$

$I = \int_0^{\pi} \frac{\sin^4 x}{(1+\cos x)^2}\, dx = \int_0^{\pi} \frac{(1-\cos^2 x)^2}{(1+\cos x)^2}\, dx$

Written by Abrar Mustafa

$$I = \int_0^{\pi} \frac{(1-\cos x)^2 (1+\cos x)^2}{(1+\cos x)^2} dx$$

$$I = \int_0^{\pi} (1-\cos x)^2 dx = \int_0^{\pi} (1-2\cos x+\cos^2 x)dx$$

$$I = \int_0^{\pi} \left(1-2\cos x + \frac{1+\cos 2x}{2}\right)dx$$

$$I = \left[x - 2\sin x + \frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi}$$

$$I = \left[\frac{3}{2}x - 2\sin x + \frac{\sin 2x}{4}\right]_0^{\pi}$$

$$I = \frac{3}{2}\pi - 0 = \frac{3\pi}{2} \text{ Ans}$$

Q No. 14:-

$$\int_0^{a} (a^2-x^2)^{\frac{5}{2}} dx.$$

Put $x = a\sin\theta$; $dx = a\cos\theta d\theta$

$$I = \int_0^{a} (a^2-x^2)^{5/2} dx$$

$$I = \int_0^{\pi/2} (a^2 - a^2\sin^2\theta)^{\frac{5}{2}} a\cos\theta d\theta$$

$$I = a^6 \int_0^{\pi/2} \cos^6\theta d\theta$$

$$I = a^6 \cdot \frac{(6-1)(6-3)(6-5)}{6.(6-2)(6-4)} \cdot \frac{\pi}{2} \quad \text{(wallis formula)}$$

$$I = a^6 \cdot \frac{5.3.1}{6.4.2} \frac{\pi}{2}$$

$$I = \frac{5a^6\pi}{32} \text{ Ans}$$

Q No. 13:-

$$\int_0^{\pi/4} \sin^4 2x dx$$

Put $2x = t$ $\quad 2dx = dt$

when $x \to 0$, $t \to 0$
$x \to \frac{\pi}{2}$, $t \to \frac{\pi}{2}$

$$I = \frac{1}{2}\int_0^{\pi/2} \sin^4 t dt = +\frac{1}{2} \frac{3 \cdot 1}{4.2} \cdot \frac{\pi}{2} = \frac{3\pi}{32} \text{ Ans}$$

Q No. 15:-

$$\int_0^{\pi/6} \sin^6 3x dx$$

Put $3x = t$, $3dx = dt$

when, $x \to 0$, $t \to 0$
$x \to \frac{\pi}{6}$, $t \to \frac{\pi}{2}$

$$I = \int_0^{\pi/6} \sin^6 3x\,dx = \int_0^{\pi/2} (\sin^6 t) \cdot \frac{1}{3}\,dt$$

$$I = \frac{1}{3}\int_0^{\pi/2} \sin^6 t\,dt$$

$$I = \frac{1}{3} \cdot \frac{(6-1)(6-3)(6-5)}{6\cdot(6-2)(6-4)} \cdot \frac{\pi}{2}$$

$$I = \frac{1}{3} \cdot \frac{5\cdot3\cdot1}{6\cdot4\cdot2} \cdot \frac{\pi}{2} = \frac{5\pi}{96} \text{ Ans}$$

QNo. 16:-

$$\int_0^{\pi/8} \sin^5 4x \cos^4 4x\,dx$$

Put $4x = z$, $dx = \frac{1}{4}dz$

when $x \to 0$, $z \to 0$

$x \to \frac{\pi}{8}$, $z \to \frac{\pi}{2}$

So, $I = \int_0^{\pi/8} \sin^5 4x \cos^4 4x\,dx = \int_0^{\pi/2} \sin^5 z \cos^4 z \cdot \frac{1}{4}\,dz$

$$I = \frac{1}{4}\int_0^{\pi/2} \sin^5 z \cdot \cos^4 z\,dz$$

Hence, P is odd and q is even so,

$$\frac{1}{4}\int_0^{\pi/2} \sin^5 z \cdot \cos^4 z\,dz = \frac{1}{4}\left[\frac{(5-1)(5-3)(4-1)(4-3)}{(5+4)(5+4-2)(5+4-4)(5+4-6)(5+4-8)}\right]$$

$$= \frac{1}{4}\left(\frac{4\cdot2\cdot3\cdot1}{9\cdot7\cdot5\cdot3\cdot1}\right) = \frac{2}{315} \text{ Ans}$$

QNo. 17:- $\int_0^{\pi/4} \cos^2 2x\,dx$

Put $2x = t$, $dx = \frac{1}{2}dt$

when, $x \to 0$, $t \to 0$

$x \to \frac{\pi}{4}$, $t \to \frac{\pi}{2}$

So, $I = \int_0^{\pi/4} \cos^2 2x\,dx = \int_0^{\pi/2} \cos^2 t \cdot \frac{1}{2}dt = \frac{1}{2}\int_0^{\pi/2} \cos^2 t\,dt$

$$I = \frac{1}{2}\left(\frac{2-1}{2} \cdot \frac{\pi}{2}\right) = \frac{\pi}{8} \text{ Ans}$$

QNo. 18:- $\int_0^{\pi/6} \cos^3 3x\,dx$

Put $3x = t$, $dx = \frac{1}{3}dt$

when, $x \to 0$, $t \to 0$

$x \to \frac{\pi}{6}$, $t \to \frac{\pi}{2}$

$$I = \int_0^{\pi/6} \cos^3 3x\,dx = \int_0^{\pi/2} \cos^3 t \cdot \frac{1}{3}\,dt = \frac{1}{3}\int_0^{\pi/2} \cos^3 t\,dt$$

$$I = +\frac{1}{3}\cdot\frac{(3-1)}{3} = \frac{2}{9} \text{ Ans.}$$

Q No. 19:- $\displaystyle\int_0^{\pi/3} \sin^2 6x \cos^4 3x\,dx$

$$I = \int_0^{\pi/3} (2\sin 3x \cos 3x)\cos^4 3x\,dx$$

$$= 4\int_0^{\pi/3} \sin^2 3x \cos^6 3x\,dx$$

$\Rightarrow$ Put $3x = z$; $dx = \frac{1}{3}dz$

when $x \to 0$, $z \to 0$

$x \to \frac{\pi}{3}$, $z \to \pi$

So $$I = \frac{4}{3}\int_0^{\pi} \sin^2 z \cos^6 z\,dz$$

$$= \frac{8}{3}\int_0^{\pi/2} \sin^2 z \cos^6 z\,dz$$

Where, $p = 2$, $q = 6$

$$I = \frac{8}{3}\left(\frac{(2-1)(6-1)(6-3)(6-5)}{(2+6)(2+6-2)(2+6-4)(2+6-6)}\cdot\frac{\pi}{2}\right)$$

$$I = \frac{8}{3}\left(\frac{1\cdot 5\cdot 3\cdot 1\cdot \pi}{8\cdot 6\cdot 4\cdot 2\cdot 2}\right) = \frac{5}{96}\pi \text{ Ans}$$

Q No. 20:- $\displaystyle\int_{\pi/3}^{\pi/2} \frac{\cos^2 x}{\sin x}\,dx$

Put $\cos x = z$

$$dz = -\sin x\,dx = \frac{\sin x}{\sin x}\cdot -\sin x\,dx$$

$$dz = -\sin^2 x\cdot\frac{dx}{\sin x}$$

$$\frac{1}{1-z^2}dz = \frac{dx}{\sin x}$$

when, $x \to \frac{\pi}{3}$, $z \to \frac{1}{2}$

$x \to \frac{\pi}{2}$, $z \to 0$

$$I = \int_{\pi/3}^{\pi/2} \frac{\cos^2 x}{\sin x}\,dx$$

$$I = \int_{1/2}^{0} z^2\cdot\left(-\frac{1}{1-z^2}\right)dz = \int_0^{1/2} \frac{z^2}{1-z^2}\,dz$$

$$I = \int_0^{1/2}\left(1 - \frac{1}{1-z^2}\right)dz$$

$$1-z^2\overline{\big)\,z^2\,}\ \ 1$$

By Partial fraction:-

$$I = \int_0^{1/2} \left[1 - \frac{1}{2}\left(\frac{1}{1+z} + \frac{1}{1-z}\right)\right] dz$$

$$I = [z]_0^{1/2} - \frac{1}{2}\left[\ln\left(\frac{1+z}{1-z}\right)\right]_0^{1/2}$$

$$I = \left(0 - \frac{1}{2}\right) - \frac{1}{2}\left[\ln 1 + \ln\left(\frac{3/2}{1/2}\right)\right]$$

$$I = -\frac{1}{2} - \left(+\frac{1}{2}\ln 3\right) = -\frac{1}{2} - \ln\sqrt{3}$$

$$I = -\frac{1}{2} \ln\sqrt{3} \text{ Ans}$$

QNo.21:-

$$\int_0^1 \frac{x^6\, dx}{\sqrt{1-x^2}}$$

Put $x = \sin\theta$, $dx = \cos\theta\, d\theta$

When, $x \to 0$, $\theta \to 0$

$x \to 1$, $\theta \to \frac{\pi}{2}$

Therefore, $$\int_0^1 \frac{x^6\, dx}{\sqrt{1-x^2}} = \int_0^{\pi/2} \frac{\sin^6\theta \cdot \cos\theta\, d\theta}{\cos\theta}$$

$$= \int_0^{\pi/2} \sin^6\theta\, d\theta$$

$$= \frac{(6-1)(6-3)(6-5)}{6\cdot(6-2)(6-4)} \cdot \frac{\pi}{2}$$

$$= \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{5\pi}{32} \text{ Ans}$$

QNo.22:-

$$\int \sec^{2n+1} x\, dx = \frac{\sec^{2n-1} x \tan x}{2n} + \left(1 - \frac{1}{2n}\right)\int \sec^{2n-1} x\, dx$$

$$I = \int \sec^{2n+1} x\, dx = \int \sec^{2n-1} x \cdot \sec^2 x\, dx$$

$$I = \sec^{2n-1} x \cdot \tan x - \int \tan x\,(2n-1)\sec^{2n-2} x \cdot \sec x \tan x\, dx$$

$$I = \sec^{2n-1} x \tan x - (2n-1)\int \sec^{2n-1} x \tan^2 x\, dx$$

$$I = \sec^{2n-1} x \tan x - (2n-1)\int \sec^{2n-1} x\,(\sec^2 x - 1)\, dx$$

$$I = \sec^{2n-1} x \tan x - (2n-1)\int \sec^{2n+1} x\, dx + (2n-1)\int \sec^{2n-1} x\, dx$$

$$(2n-1) I + I = \sec^{2n-1} x \tan x + (2n-1)\int \sec^{2n-1} x\, dx$$

$$I = \frac{\sec^{2n-1} x \tan x}{2n} + \left(1 - \frac{1}{2n}\right)\int \sec^{2n-1} x\, dx$$

Hence Proved

Q No. 23:- Obtain reduction formula $\int \frac{dx}{(a^2+x^2)^n}$, n is an Integer.
Also, Show That: $\int_0^\infty \frac{dx}{(1+x^2)^5} = \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{\pi}{2}$

Sol:-

$$\int \frac{dx}{(a^2+x^2)^n} = \int (a^2+x^2)^{-n}\cdot 1\, dx$$

$$= (a^2+x^2)^{-n}\cdot x - \int -n(a^2+x^2)^{-n-1}\cdot 2x \cdot x\, dx$$

$$= x(a^2+x^2)^{-n} + \int 2n x^2 (a^2+x^2)^{-n-1} dx$$

$$= x(a^2+x^2)^{-n} + 2n\int x^2 (a^2+x^2)^{-n-1} dx$$

$$= x(a^2+x^2)^{-n} + 2n\int (x^2+a^2-a^2)(a^2+x^2)^{-n-1} dx$$

$$= x(a^2+x^2)^{-n} + 2n\int (a^2+x^2)^{-n} dx - 2na^2\int (a^2+x^2)^{-n-1} dx.$$

$$\Rightarrow 2na^2\int \frac{dx}{(a^2+x^2)^{n+1}} = x(a^2+x^2)^{-n} + (2n-1)\int (a^2+x^2)^{-n} dx$$

Replace n into n-1, we have,

$$2(n-1)a^2\int \frac{dx}{(a^2+x^2)^n} = \frac{x}{(a^2+x^2)^{n-1}} + (2n-3)\int \frac{dx}{(a^2+x^2)^{n-1}}$$

$$\int \frac{dx}{(a^2+x^2)^n} = \frac{x}{2(n-1)a^2(a^2+x^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)}\int \frac{dx}{(a^2+x^2)^{n-1}}$$

Integrating b/w the Limit 0 to ∞.

$$\int_0^\infty \frac{dx}{(a^2+x^2)^n} = \frac{2n-3}{2a^2(n-1)}\int_0^\infty \frac{dx}{(a^2+x^2)^{n-1}}$$

$$\Rightarrow \int_0^\infty \frac{dx}{(1+x^2)^5} = \frac{7}{2\cdot 4}\int_0^\infty \frac{dx}{(1+x^2)^4}$$

$$\Rightarrow \int_0^\infty \frac{dx}{(1+x^2)^4} = \frac{5}{2\cdot 3}\int_0^\infty \frac{dx}{(1+x^2)^3}$$

$$\Rightarrow \int_0^\infty \frac{dx}{(1+x^2)^3} = \frac{3}{2\cdot 2}\int_0^\infty \frac{dx}{(1+x^2)^2}$$

$$\int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{1}{2\cdot 1}\int_0^\infty \frac{dx}{(1+x^2)} = \frac{1}{2\cdot 1}\left|\tan^{-1}x\right|_0^\infty$$

$$= \frac{1}{2} \cdot \frac{\pi}{2}$$

We have,

$$\int_0^\infty \frac{dx}{(1+x^2)^5} = \frac{7\cdot5\cdot3\cdot1}{8\cdot6\cdot4\cdot2} \cdot \frac{\pi}{2} = \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8} \cdot \frac{\pi}{2}$$

Hence, Proved

QNo. 24:-

$$I_n = \int_0^1 x^p (1-x^q)^n dx$$

Prove :- $(qn+p+1) I_n = qn I_{n-1}$

$$I = \int x^p (1-x^q)^n dx = (1-x^q)^n \frac{x^{p+1}}{p+1} - \int \frac{x^{p+1}}{p+1} \cdot n(1-x^q)^{n-1} \cdot (-q x^{q-1}) dx$$

$$= \frac{x^{p+1}(1-x^q)^n}{p+1} + \frac{qn}{p+1} \int x^{p+1} \cdot x^{q-1} (1-x^q)^{n-1} dx$$

$$= \frac{x^{p+1}(1-x^q)^n}{p+1} + \frac{qn}{p+1} \int x^{p+q} (1-x^q)^{n-1} dx$$

$$= \frac{x^{p+1}(1-x^q)^n}{p+1} - \frac{qn}{p+1} \int x^p (1-x^q+1)(1-x^q)^{n-1} dx$$

$$= \frac{x^{p+1}(1-x^q)^n}{p+1} - \frac{qn}{p+1} \int x^p (1-x^q)^n + \frac{qn}{p+1} \int x^p (1-x^q)^{n-1} dx$$

So,

$$\left(1+\frac{qn}{p+1}\right) \int x^p (1-x^q)^n dx = \frac{x^{p+1}(1-x^q)^n}{p+1} + \frac{qn}{p+1} \int x^p (1-x^q)^{n-1} dx$$

Now,

$$\left(\frac{qn+p+1}{p+1}\right) \int_0^1 x^p (1-x^q)^n dx = \frac{x^{p+1}(1-x^q)^n}{p+1} \Big|_0^1 + \frac{qn}{p+1} \int_0^1 x^p (1-x^q)^{n-1} dx$$

$$(qn+p+1) I_n = qn \, I_{n-1}$$

Hen Proved.

Evaluate I_n : n is +ve integer:-

$$I_n = \frac{qn}{(qn+p+1)} I_{n-1}$$

$$I_{n-1} = I_{n-2} \, q(n-1) / q(n-1)+p+1$$

$$I_{n-2} = \frac{q(n-2)}{q(n-2)+p+1} I_{n-3}$$

$$I_{n-3} = \frac{q(n-3)}{q(n-3)+p+1} I_{n-4}$$

$$\vdots$$

$$I_3 = \frac{3q}{3q+p+1} I_2$$

$$I_2 = \frac{2q}{2q+p+1} I_1$$

$$I_1 = \frac{q}{1q+p+1} I_0$$

Now, $I_0 = \int_0^1 x^p\,dx = \left.\frac{x^{p+1}}{p+1}\right|_0^1 = \frac{1}{p+1} - 0 = \frac{1}{p+1}$

We get:-

$$I_n = \frac{qn\; q(n-1)\; q(n-2)\; q(n-3) \cdots 3q \cdot 2q \cdot 1q \cdot 1}{(qn+p+1)(q(n-1)+p+1)(q(n-2)+p+1)\cdots(q+p+1)(p+1)}$$

$$I_n = \frac{q^n\, n!}{(qn+p+1)(q(n-1)+p+1)(q(n-2)+p+1)\cdots(q+p+1)(p+1)}$$

which is required.

Q No. 25:-

Obtain a reduction formula for $\int \frac{x^n}{\sqrt{1-x^2}}\,dx$.

$$I = \int \frac{x^n}{\sqrt{1-x^2}}\,dx = \int x^n (1-x^2)^{-\frac{1}{2}} \frac{(-2x)}{(-2x)}\,dx$$

$$= -\frac{1}{2}\int x^{n-1}(1-x^2)^{-\frac{1}{2}}(-2x)\,dx$$

$$= -\frac{1}{2}\left\{ x^{n-1}\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - \int (n-1)x^{n-2}\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\,dx \right\}$$

$$I = -\frac{1}{2}\left\{ x^{n-1}\frac{\sqrt{1-x^2}}{1/2} - \int (n-1)x^{n-2}\frac{\sqrt{1-x^2}}{\frac{1}{2}}\,dx \right\}$$

$$= -x^{n-1}\sqrt{1-x^2} + (n-1)\int x^{n-2}(1-x^2)^{-\frac{1}{2}+1}\,dx$$

$$= -x^{n-1}\sqrt{1-x^2} + (n-1)\int x^{n-2}(1-x^2)^{-1/2}\cdot(1-x^2)\,dx$$

$$I = -x^{n-1}\sqrt{1-x^2} + (n-1)\int x^{n-2}(1-x^2)^{-1/2} - (n-1)\int \frac{x^n}{\sqrt{1-x^2}}\,dx$$

$$(1+(n-1))\,I = -x^{n-1}\sqrt{1-x^2} + (n-1)\int \frac{x^{n-2}}{\sqrt{1-x^2}}\,dx$$

$$nI = -x^{n-1}\sqrt{1-x^2} + n-1\int \frac{x^{n-2}}{\sqrt{1-x^2}}dx$$

$$I = \frac{-x^{n-1}\sqrt{1-x^2}}{n} + \frac{n-1}{n}\int \frac{x^{n-2}}{\sqrt{1-x^2}}dx$$

Evaluate :- $\int \frac{x^3}{\sqrt{1-x^2}}dx$

Put $n=3$ in above eq.

$$I = \int \frac{x^3}{\sqrt{1-x^2}}dx = \frac{-x^2\sqrt{1-x^2}}{3} + \frac{2}{3}\int \frac{x}{\sqrt{1-x^2}}dx$$

$$I = \frac{-x^2\sqrt{1-x^2}}{3} + \frac{2}{3(-2)}\int \frac{-2x}{\sqrt{1-x^2}}dx$$

$$I = \frac{-x^2\sqrt{1-x^2}}{3} + \left(-\frac{1}{3}\right)\frac{\sqrt{1-x^2}}{1/2}$$

$$I = \frac{-x^2\sqrt{1-x^2}}{3} - \frac{2}{3}\sqrt{1-x^2}$$

Ans

Q No. 26:- Calculate value of $\int_0^{2a} x^n\sqrt{2ax-x^2}\,dx$, n is +ve integer.

Sol:-

$$I' \Rightarrow \int x^n\sqrt{2ax-x^2}\,dx = \int x^n\sqrt{x}\sqrt{2a-x}\,dx$$

$$\int x^n\sqrt{2ax-x^2}\,dx = \int x^{n+\frac{1}{2}}\sqrt{2a-x}\,dx = -\int x^{n+\frac{1}{2}}\sqrt{2a-x}\,(-1)dx$$

$$I' = -\left\{x^{n+\frac{1}{2}}\frac{(2a-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \int \left(n+\frac{1}{2}\right)x^{n+\frac{1}{2}-1}\cdot\frac{(2a-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}dx\right\}$$

$$I' = -\left\{x^{n+\frac{1}{2}}\frac{2}{3}(2a-x)^{3/2} - \frac{2n+1}{3}\int x^{n+\frac{1}{2}-1}(2a-x)^{1/2}(2a-x)dx\right\}$$

$$I' = \frac{-2}{3}x^{n+1/2}(2a-x)^{3/2} + \frac{2n+1}{3}\int 2ax^{n+\frac{1}{2}-1}(2a-x)^{1/2} - x^{n+1/2}(2a-x)^{1/2}dx$$

$$I' = -\frac{2}{3}x^{n+\frac{1}{2}}(2a-x)^{3/2} + \frac{2n+1}{3}\int\left(2ax^{n+\frac{1}{2}-1}(2a-x)^{1/2} - x^{n+\frac{1}{2}}(2a-x)^{1/2}\right)dx$$

$$\left(1+\frac{2n+1}{3}\right)I' = -\frac{2}{3}x^{n+\frac{1}{2}}(2a-x)^{3/2} + \frac{2(2n+1)}{3}\int ax^{n+\frac{1}{2}-1}(2a-x)^{1/2}dx$$

$$(2n+4)I' = -2x^{n+\frac{1}{2}}(2a-x)^{3/2} + 2(2n+1)\int ax^{n+\frac{1}{2}-1}(2a-x)^{1/2}dx$$

$$I' = \frac{-x^{n+\frac{1}{2}}(2a-x)^{\frac{3}{2}}}{(n+2)} + \frac{(2n+1)}{n+2}\int ax^{n+\frac{1}{2}-1}\sqrt{2a-x}\,dx$$

Therefore,

$$I_n = a\cdot\frac{2n+1}{n+2}\int_0^{2a} x^{n+\frac{1}{2}-1}\sqrt{2a-x}\,dx$$

So,

$$I_n = \frac{2n+1}{n+2} a I_{n-1}$$

$$I_{n-1} = \frac{2(n-1)+1}{(n-1)+2} a I_{n-2} = \frac{(2n-1)a}{n+1} I_{n-2}$$

$$I_{n-2} = \frac{(2n-3)a}{n} I_{n-3}$$

$$\vdots$$

$$I_3 = \frac{7a}{5} I_2 \quad ; \quad I_2 = \frac{5a}{4} I_1$$

$$I_1 = \frac{3a}{3} I_0$$

Now, $I_0 = \int_0^{2a} x^{1/2} \sqrt{2a-x}\, dx$

$\Rightarrow$ Put $x = 2a\sin^2\theta$; $dx = 4a\sin\theta\cos\theta\, d\theta$

$$I_0 = \int_0^{\pi/2} \sqrt{2ax - x^2}\, dx$$

$$I_0 = \int_0^{\pi/2} \sqrt{4a^2\sin^2\theta - 4a^2\sin^4\theta}\, . (4a\sin\theta\cos\theta)\, d\theta$$

$$I_0 = \int_0^{\pi/2} \sqrt{4a^2\sin^2\theta . \cos^2\theta}\, . 4a\sin\theta\cos\theta\, d\theta$$

$$I_0 = \int_0^{\pi/2} 2a\sin\theta . \cos\theta\, 4a\sin\theta\cos\theta\, d\theta$$

$$I_0 = \int_0^{\pi/2} 8a^2 \sin^2\theta\cos^2\theta\, d\theta$$

$$I_0 = 8a^2 \int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta$$

$$I_0 = 8a^2 \, \frac{2 . 1 . \pi}{4 . 2 . 2} = \frac{a^2\pi}{2}$$

So, we have:-

$$I_n = \frac{(2n+1)(2n-1) \cdots 5 . 3}{(n+2)(n+1) n \cdots 4 . 3} a^n . \frac{a^2 \pi}{2}$$

$$I_n = \frac{(2n+1)(2n-1) \cdots 5 . 3}{(n+2)(n+1) \cdots 4 . 3} a^{n+2} . \frac{\pi}{2} \quad (A)$$

i):- Evaluate:-

$$I_1 = \int_0^{2a} x \sqrt{2a - x^2}\, dx$$

There are two ways to solve this I_1 and also I_2 (i) Put, $x = 2a\sin^2\theta\, d\theta$

$\Rightarrow$ (ii):- Put $n = 1$ in eq (A):-

$$I_1 = \frac{3}{3} a^3 \frac{\pi}{2} = \frac{a^3 \pi}{2}$$

ii) :- $I_4 = \int_0^{2a} x^4 \sqrt{2ax - x^2}\, dx$

we use eq. (A):-

Put $n = 4$ in (A).

$I_4 := \frac{9.7.5.3}{6.5.4.3} a^6 \frac{\pi}{2} = \frac{21 a^6 \pi}{16}$ Ans

Q No. 27:- $I_n = \int x^n (a^2 - x^2)^{1/2} dx$

Prove that:-

$$I_n = \frac{x^{n-1}(a^2 - x^2)^{3/2}}{n+2} + \frac{n-1}{n+2} a^2 I_{n-2}$$

Sol:-

$$I_n = \int x^n (a^2 - x^2)^{1/2} dx = -\frac{1}{2}\int x^{n-1} (a^2 - x^2)^{1/2} (-2x) dx$$

$$= -\frac{1}{2}\left\{ x^{n-1} \cdot \frac{(a^2 - x^2)^{3/2}}{3/2} - \int (n-1) x^{n-2} \cdot \frac{(a^2 - x^2)^{3/2}}{3/2} dx \right\}$$

$$I_n = -\frac{1}{2}\left\{ \frac{2}{3} x^{n-1} (a^2 - x^2)^{3/2} - \int \frac{2(n-1)}{3} x^{n-2} (a^2 - x^2)^{3/2} dx \right\}$$

$$I_n = -\frac{1}{3} x^{n-1} (a^2 - x^2)^{3/2} + \frac{(n-1)}{3} \int x^{n-2} (a^2 - x^2)^{1/2} (a^2 - x^2) dx$$

$$I_n = -\frac{1}{3} x^{n-1} (a^2 - x^2)^{3/2} + \frac{n-1}{3} \int a^2 x^{n-2} (a^2 - x^2)^{1/2} - x^n \sqrt{a^2 - x^2}\, dx$$

$$\left(1 + \frac{n-1}{3}\right) I_n = -\frac{x^{n-1}(a^2 - x^2)^{3/2}}{3} + \frac{(n-1)a^2}{3} \int x^{n-2} \sqrt{a^2 - x^2}\, dx$$

$$\Rightarrow I_n = \frac{-x^{n-1}(a^2 - x^2)^{3/2}}{n+2} + \frac{(n-1)}{n+2} a^2 I_{n-2}$$

Hence Proved

Evaluate:- $\int_0^a x^4 \sqrt{a^2 - x^2}\, dx$

$I_4 = \int_0^a x^4 \sqrt{a^2 - x^2}\, dx$

$$\Rightarrow \int x^4 \sqrt{a^2 - x^2}\, dx = \frac{-x^3 (a^2 - x^2)^{3/2}}{6} + \frac{3}{6} a^2 I_{4-2}$$

Now,

$$\int x^2 \sqrt{a^2 - x^2}\, dx = \frac{-x (a^2 - x^2)^{3/2}}{4} + \frac{1}{4} a^2 I_0$$

So,

$$I_4 = \int_0^a x^4 \sqrt{a^2 - x^2}\, dx = \frac{1}{2} a^2 \cdot \frac{1}{4} a^2 \int_0^a x^0 (a^2 - x^2)^{1/2} dx$$

$$I_4 = \frac{a^4}{8} \int_0^a \sqrt{a^2 - x^2}\, dx$$

$$\Rightarrow I_4 = \frac{a^4}{8}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$I_4 = \frac{a^4}{8}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi a^6}{32} \quad \text{Ans}$$

Q No. 28:-

$$\int x^m (\ln x)^n dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m (\ln x)^{n-1} dx$$

Sol:-

$$\int x^m (\ln x)^n dx = (\ln x)^n \cdot \frac{x^{m+1}}{m+1} - \int n(\ln x)^{n-1}\cdot\frac{1}{x}\cdot\frac{x^{m+1}}{m+1} dx$$

$$= \frac{(\ln x)^n x^{m+1}}{m+1} - \int \frac{n}{m+1}(\ln x)^{n-1} x^m dx$$

$$= \frac{(\ln x)^n x^{m+1}}{m+1} - \frac{n}{m+1}\int x^m (\ln x)^{n-1} dx \qquad (A)$$

which is required.

i):- $\int x^m (\ln x)^3 dx$

$$\int x^m (\ln x)^3 dx = \frac{x^{m+1}(\ln x)^3}{m+1} - \frac{3}{m+1}\int x^m (\ln x)^2 dx$$

$$\Rightarrow \int x^m (\ln x)^2 dx = \frac{x^{m+1}(\ln x)^2}{m+1} - \frac{2}{m+1}\int x^m (\ln x)^1 dx$$

$$\Rightarrow \int x^m (\ln x) dx = \frac{x^{m+1}(\ln x)}{m+1} - \frac{1}{m+1}\int x^m dx$$

$$= \frac{x^{m+1}(\ln x)}{m+1} - \frac{x^{m+1}}{(m+1)^2}$$

we have

$$\int x^m (\ln x)^3 dx = \frac{x^{m+1}(\ln x)^3}{m+1} - \frac{3x^{m+1}(\ln x)^2}{(m+1)^2} + \frac{6x^{m+1}\ln x}{(m+1)^3} - \frac{6x^{m+1}}{(m+1)^4}$$

Ans

ii):- $\int_0^1 x^m (\ln x)^n dx$

(A) $\Rightarrow$ we have :- $\int_0^1 x^m (\ln x)^n dx = -\frac{n}{m+1}\int_0^1 x^m (\ln x)^{n-1} dx$

$$= \frac{-n}{m+1} I_{m,n-1}$$

So,

$$I_{m,n} = \frac{-n}{m+1} I_{m,n-1}$$

$$\Rightarrow I_{m,n-1} = -\frac{n-1}{m+1} I_{m,n-2}$$

$$\vdots$$

$$I_{m,2} = -\frac{2}{m+1} I_{m,1}$$

$$I_{m,1} = -\frac{1}{m+1} I_{m,0} = \frac{-1}{m+1}\cdot\frac{1}{m+1}$$

Therefore,

$$I_{m,n} = \frac{(-1)^n n!}{(m+n)^{n+1}} \quad \text{Ans}$$

Q No. 29:- $\int \cos^m x \sin nx\, dx$

$$I=\int \cos^m x \sin nx\, dx = \cos^m x\left(-\frac{\cos nx}{n}\right) - \int m\cos^{m-1}x\,(-\sin x)\left(\frac{-\cos nx}{n}\right)dx$$

$$I = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1}x \sin x \cos nx\, dx$$

Since, $\sin(n-1)x = \sin(nx-x) = \sin nx\cos x - \cos nx \sin x$

$$\Rightarrow \cos nx \sin x = \sin nx \cos x - \sin(n-1)x$$

Therefore:-

$$I=\int \cos^m x \sin nx\, dx = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1}x\,(\sin nx \cos x - \sin(n-1)x$$

$$\left(1+\frac{m}{n}\right)I = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n}\int \cos^{m-1}x \sin(n-1)x\, dx\ ]$$

$$(m+n)I = -\cos^m x \cos nx + m\int \cos^{m-1}x \sin(n-1)x\, dx$$

$$I = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n}\int \cos^{m-1}x \sin(n-1)x\, dx$$

Ans

Now

$$\int_0^{\pi/2} \cos^m x \sin nx\, dx = \frac{1}{m+n} + \frac{m}{m+n}\int_0^{\pi/2} \cos^{m-1} \sin(n-1)x\, dx$$

Put $m=5$, $n=3$

$$\int_0^{\pi/2} \cos^5 x \sin 3x\, dx = \frac{1}{8} + \frac{5}{8}\int_0^{\pi/2} \cos^4 x \sin 2x\, dx$$

$$\Rightarrow \int_0^{\pi/2} \cos^4 x \sin 2x\, dx = \frac{1}{6} + \frac{4}{6}\int_0^{\pi/2} \cos^3 x \sin x\, dx$$

$$\Rightarrow \int_0^{\pi/2} \cos^3 x \sin x\, dx = \frac{1}{4} + \frac{3}{4}\int_0^{\pi/2} \cos^2 x$$

$$= \frac{1}{4} + \frac{3}{4}\int_0^{\pi/2} \frac{1+\cos 2x}{2}\, dx$$

$$= \frac{1}{4} + \frac{3}{4}\left(\frac{x}{2} + \frac{\sin 2x}{4}\right)_0^{\pi/2}$$

$$= \frac{1}{4} + \frac{3}{4}($$

$$\Rightarrow \int_0^{\pi/2} \cos^3 x \sin x\, dx = \frac{1}{4} + \frac{3}{4}\int_0^{\pi/2} \cos^2 x\,(\sin(1-1)x)\, dx$$

$$= \frac{1}{4} + \frac{3}{4}\cdot 0 = \frac{1}{4}$$

So,

$$\int_0^{\pi/2} \cos^5 x \sin 3x\, dx = \frac{1}{8} + \frac{5}{48} + \frac{20}{48\cdot 4} = \frac{24+40}{192} = \frac{64}{192}$$

$= \frac{1}{3}$ Ans

Q No. 30:- Find a reduction formula for $\int \frac{x^n}{\sqrt{ax^2+2bx+c}}\,dx$

Sol:- $\int \frac{x^n}{\sqrt{ax^2+2bx+c}}\,dx \Rightarrow I = \int x^n (ax^2+2bx+c)^{-\frac{1}{2}}\,dx$

$$= \int x^n (ax^2+2bx+c)^{-\frac{1}{2}}\,dx \cdot \frac{2ax}{2ax} + 2b - 2b$$

So $I = \frac{1}{2a}\int x^{n-1}(ax^2+2bx+c)^{-\frac{1}{2}}(2ax+2b-2b)\,dx$

$$I = \frac{1}{2a}\int x^{n-1}(ax^2+2bx+c)^{-\frac{1}{2}}(2ax+2b)\,dx - \frac{2b}{2a}\int x^{n-1}(ax^2+2bx+c)^{-\frac{1}{2}}\,dx$$

By first Integral:-

$$\Rightarrow I_1 = \frac{1}{2a}\left\{ x^{n-1}\frac{(ax^2+2bx+c)^{\frac{1}{2}}}{1/2} - \int (n-1)x^{n-2}\cdot\frac{(ax^2+2bx+c)^{1/2}}{1/2}\,dx\right.$$

$$I_1 = \frac{x^{n-1}(ax^2+2bx+c)^{1/2}}{a} - \frac{n-1}{a}\int x^{n-2}\cdot(ax^2+2bx+c)^{-\frac{1}{2}+1}\,dx$$

$$I_1 = \frac{x^{n-1}(ax^2+2bx+c)^{1/2}}{a} - \frac{n-1}{a}\int x^{n-2}(ax^2+2bx+c)^{-\frac{1}{2}}\cdot ax^2\,dx$$

$$- \frac{n-1}{a}2b\int x^{n-1}(ax^2+2bx+c)^{-\frac{1}{2}}\,dx - \frac{n-1}{a}c\int x^{n-2}(ax^2+2bx+c)^{-1/2}\,dx$$

$$I_1 = \frac{x^{n-1}\sqrt{ax^2+2bx+c}}{a} - n-1\int \frac{x^n}{\sqrt{ax^2+2bx+c}}\,dx - \frac{2(n-1)b}{a}\int \frac{x^{n-1}}{\sqrt{ax^2+2bx+c}}\,dx$$

$$- \frac{(n-1)c}{a}\int \frac{x^{n-2}}{\sqrt{ax^2+2bx+c}}\,dx.$$

We get:-

$$I(1+n-1) = \frac{x^{n-1}\sqrt{ax^2+2bx+c}}{a} - \frac{(2n-2)b}{a}\int \frac{x^{n-1}}{\sqrt{ax^2+2bx+c}}\,dx - \frac{(n-1)c}{a}$$

$$\int \frac{x^{n-2}}{\sqrt{ax^2+2bx+c}}\,dx - \frac{b}{a}\int x^{n-1}(ax^2+2bx+c)^{-\frac{1}{2}}\,dx$$

$$I = \frac{x^{n-1}\sqrt{ax^2+2bx+c}}{a} - \frac{b}{an}(2n-1)\int \frac{x^{n-1}}{\sqrt{ax^2+2bx+c}}\,dx$$

$$- \frac{(n-1)c}{an}\int \frac{x^{n-2}}{\sqrt{ax^2+2bx+c}}\,dx$$

Ans